

# Generalized feedback vertex set problems on bounded-treewidth graphs: chordality is the key to single-exponential parameterized algorithms\*

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## Abstract

It has long been known that FEEDBACK VERTEX SET can be solved in time  $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$  on graphs of treewidth  $w$ , but it was only recently that this running time was improved to  $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$ , that is, to single-exponential parameterized by treewidth. We investigate which generalizations of FEEDBACK VERTEX SET can be solved in a similar running time. Formally, for a class of graphs  $\mathcal{P}$ , the BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION problem asks, given a graph  $G$  on  $n$  vertices and positive integers  $k$  and  $d$ , whether  $G$  contains a set  $S$  of at most  $k$  vertices such that each block of  $G - S$  has at most  $d$  vertices and is in  $\mathcal{P}$ . Assuming that  $\mathcal{P}$  is recognizable in polynomial time and satisfies a certain natural hereditary condition, we give a sharp characterization of when single-exponential parameterized algorithms are possible for fixed values of  $d$ :

- if  $\mathcal{P}$  consists only of chordal graphs, then the problem can be solved in time  $2^{\mathcal{O}(wd^2)} n^{\mathcal{O}(1)}$ ,
- if  $\mathcal{P}$  contains a graph with an induced cycle of length  $\ell \geq 4$ , then the problem is not solvable in time  $2^{o(w \log w)} n^{\mathcal{O}(1)}$  even for fixed  $d = \ell$ , unless the ETH fails.

We also study a similar problem, called BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION, where the target graphs have connected components of small size rather than blocks of small size, and we present analogous results. For this problem, we also show that if  $d$  is part of the input and  $\mathcal{P}$  contains all chordal graphs, then it cannot be solved in time  $f(w)n^{o(w)}$  for some function  $f$ , unless the ETH fails.

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# 1 Introduction

Treewidth is a measure of how well a graph accommodates a decomposition into a tree-like structure. In the field of parameterized complexity, many NP-hard problems have been shown to have FPT algorithms when parameterized by treewidth; for example, COLORING, VERTEX COVER, FEEDBACK VERTEX SET, and STEINER TREE (see [5, Section 7] for further examples). In fact, Courcelle [4] established a meta-theorem that every problem definable in  $\text{MSO}_2$  logic can be solved in linear time on graphs of bounded treewidth. While Courcelle's Theorem is a very general tool for obtaining algorithmic results, for specific problems dynamic programming techniques usually give algorithms where the running time  $f(w)n^{\mathcal{O}(1)}$  has better dependence on treewidth  $w$ . There is some evidence that careful implementation of dynamic programming (plus maybe some additional ideas) gives optimal dependence for some problems (see, e.g., [11]).

For FEEDBACK VERTEX SET, standard dynamic programming techniques give  $2^{\mathcal{O}(w \log w)} n^{\mathcal{O}(1)}$ -time algorithms and it was considered plausible that this could be the best possible running time. Hence, it was a remarkable surprise when it turned out that  $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$  algorithms are also possible for this problem by various techniques: Cygan et al. [6] obtained a  $3^w n^{\mathcal{O}(1)}$ -time randomized algorithm by using the so-called Cut & Count technique, and Bodlaender et al. [1] showed there is a deterministic  $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$ -time algorithm by using a rank-based approach and the concept of representative sets. This was also later shown in the more general setting of representative sets in matroids by Fomin et al. [10].

**Generalized feedback vertex set problems.** In this paper, we explore the extent to which these results apply for generalizations of FEEDBACK VERTEX SET. The FEEDBACK VERTEX SET problem asks for a set  $S$  of at most  $k$  vertices such that  $G - S$  is acyclic, or in other words,  $G - S$  has only *trivial* blocks (that is, they consist of a single edge or vertex). We consider generalizations where we allow the blocks to be some other type of small graph, such as triangles, small cycles, or small cliques; these generalizations were first studied in [3]. The main result of this paper is that the existence of single-exponential algorithms for such a problem is closely linked to whether the small graphs we are allowing are all chordal or not. Formally, we consider the following problem:

BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION

**Parameter:**  $d, w$

**Input:** A graph  $G$  of treewidth at most  $w$ , and positive integers  $d$  and  $k$ .

**Question:** Is there a set  $S$  of at most  $k$  vertices in  $G$  such that each block of  $G - S$  has at most  $d$  vertices and is in  $\mathcal{P}$ ?

The result of Bodlaender et al. [1] implies that when  $d = 2$ , BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION can be solved in time  $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$ . Our main question is for which graph classes  $\mathcal{P}$  can this problem be solved in time  $2^{\mathcal{O}(w)} n^{\mathcal{O}(1)}$ , when we regard  $d$  as a fixed constant. It turns out that chordal graphs have an important role in answering this question. A graph is *chordal* if it has no induced cycles of length at least 4. We show that if  $\mathcal{P}$  consists of only chordal graphs, then we can solve this problem in single-exponential time for fixed  $d$ .

**Theorem 1.1.** *Let  $\mathcal{P}$  be a class of graphs that is block-hereditary, recognizable in polynomial time, and consists of only chordal graphs. Then BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION can be solved in time  $2^{\mathcal{O}(wd^2)} k^2 n$  on graphs with  $n$  vertices and treewidth  $w$ .*

The condition that  $\mathcal{P}$  is block-hereditary ensures that the class of graphs with blocks in  $\mathcal{P}$  is hereditary; a formal definition is given in Section 2. We complement this result by showing that

if  $\mathcal{P}$  contains a graph that is not chordal, then single-exponential algorithms are not possible (assuming ETH), even for fixed  $d$ . Note that if  $\mathcal{P}$  is block-hereditary and contains a graph that is not chordal, then this graph contains a chordless cycle on  $\ell \geq 4$  vertices, and consequently the cycle graph on  $\ell$  vertices is also in  $\mathcal{P}$ .

**Theorem 1.2.** *If  $\mathcal{P}$  contains the cycle graph on  $\ell \geq 4$  vertices, then BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION is not solvable in time  $2^{o(w \log w)} n^{\mathcal{O}(1)}$  on graphs of treewidth at most  $w$  even for fixed  $d = \ell$ , unless the ETH fails.*

**Bounded-size components.** Using a similar technique, we can obtain analogous results for a slightly simpler problem, which we call BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION, where we want to remove at most  $k$  vertices such that each connected component of the resulting graph has at most  $d$  vertices and belongs to  $\mathcal{P}$ . If we have only the size constraint (i.e.,  $\mathcal{P}$  contains every graph), then this problem is known as COMPONENT ORDER CONNECTIVITY [7].

BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION **Parameter:**  $d, w$   
**Input:** A graph  $G$  of treewidth at most  $w$ , and positive integers  $d$  and  $k$ .  
**Question:** Is there a set  $S \subseteq V(G)$  with  $|S| \leq k$  such that each connected component of  $G - S$  has at most  $d$  vertices and is in  $\mathcal{P}$ ?

Drange, Dregi, and van 't Hof [7] studied the parameterized complexity of a weighted variant of the COMPONENT ORDER CONNECTIVITY problem; their results imply, in particular, that COMPONENT ORDER CONNECTIVITY can be solved in time  $2^{\mathcal{O}(k \log d)} n$ , but is  $W[1]$ -hard parameterized by only  $k$  or  $d$ . The corresponding edge-deletion problem, parameterized by treewidth, was studied by Enright and Meeks [8]. For general classes  $\mathcal{P}$ , we prove results that are analogous to those for BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION.

**Theorem 1.3.** *Let  $\mathcal{P}$  be a class of graphs that is hereditary, recognizable in polynomial time, and consists of only chordal graphs. Then BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION can be solved in time  $2^{\mathcal{O}(wd^2)} k^2 n$  on graphs with  $n$  vertices and treewidth  $w$ .*

**Theorem 1.4.** *If  $\mathcal{P}$  contains the cycle graph on  $\ell \geq 4$  vertices, then BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION is not solvable in time  $2^{o(w \log w)} n^{\mathcal{O}(1)}$  on graphs of treewidth at most  $w$  even for fixed  $d = \ell$ , unless the ETH fails.*

When  $d$  is not fixed, one might ask whether BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION admits an  $f(w)n^{\mathcal{O}(1)}$ -time algorithm; that is, an FPT algorithm parameterized only by treewidth. We provide a negative answer, showing that the problem is  $W[1]$ -hard when  $\mathcal{P}$  contains all chordal graphs, even parameterized by both treewidth and  $k$ . We further prove two stronger lower bound results assuming the ETH holds.

**Theorem 1.5.** *Let  $\mathcal{P}$  be a hereditary class containing all chordal graphs. Then BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION is  $W[1]$ -hard parameterized by the combined parameter  $(w, k)$ . Moreover, unless the ETH fails, this problem*

1. *has no  $f(w)n^{o(w)}$ -time algorithm; and*
2. *has no  $f(k')n^{o(k'/\log k')}$ -time algorithm, where  $k' = w + k$ .*

**Techniques.** A pair  $(G, S)$  consisting of a graph  $G$  and a vertex subset  $S$  of  $G$  will be called a boundaried graph, and an  $S$ -block of  $G$  is a block of  $G$  containing an edge in  $S$ . The algorithm for BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION uses several lemmas on  $S$ -blocks of boundaried graphs  $(G, S)$ , which appear in Section 3. The key property is the following:

- (\*) when we merge two boundaried graphs  $(G, S)$  and  $(H, S)$  into a graph  $G'$ , to decide whether each  $S$ -block of  $G'$  is some fixed target graph that is chordal, it is sufficient to know, for each non-trivial block  $B$  of  $G[S]$  or  $H[S]$ , some local information about  $B$  in the  $S$ -block containing  $B$  in  $G$  or  $H$ , respectively.

We think of target graphs as labeled graphs where any two vertices in the same block have distinct labels in  $\{1, \dots, d\}$ , and the local information referred to in (\*) is the set of labels of neighbors of  $B$  in the  $S$ -block containing  $B$ . The related result is stated as Proposition 3.1. This will be used to determine whether each of the  $S$ -blocks of  $G'$  is one of the target graphs in  $\mathcal{P}$ . After then, to decide whether  $G'$  is a desired graph, it remains to check that the whole graph has no chordless cycle, since there is a possibility of linking two controlled blocks by a sequence of uncontrolled blocks in both  $G$  and  $H$ , and thus creating a chordless cycle in  $G'$ . This second part can be dealt with in a similar manner to the single-exponential time algorithm for FEEDBACK VERTEX SET, using representative-set techniques.

**Lower bounds.** Theorem 1.4 is obtained by a reduction from the problem of finding an independent set of size  $k$  in a graph on  $k^2$  vertices, hence  $O(k^4)$  edges. One can think of those vertices as forming a  $k$ -by- $k$  grid, where one should select exactly one vertex per row and per column. This problem cannot be solved in time  $2^{o(k \log k)} k^{\mathcal{O}(1)}$ , unless the ETH fails [13]. The crucial point is that the treewidth of the equivalent instances of BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION and BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION should be in  $\Theta(k)$ . We achieve this by stretching the information into a chain of  $O(k^4)$  almost identical pieces, each encoding one edge of the initial graph. The pieces are linked by small separators of size  $2k$  that propagate the row and column indices of each of the  $k$  choices for the independent set.

For Theorem 1.5, we propose a reduction from MULTICOLORED CLIQUE for the first item, and more or less the same reduction but from SUBGRAPH ISOMORPHISM for the second. Again, the crux of the construction is obtaining an instance with low treewidth. This time, we rely on an injective mapping of edges into integers, which is a folklore trick. Vertices of the initial graph are encoded as a collection of candidate places where the constructed graph can be disconnected, regularly positioned on two *paths*, one with a small weight and one with a larger weight. The edge gadget is similarly realized with certain vertices that are candidates for removal, as they can disconnect the constructed graph, each corresponding to a specific edge.

**Organization.** The paper is organized as follows. Section 2 introduces the necessary notions including labelings, treewidth, and boundaried graphs. In Section 3, we prove structural lemmas about  $S$ -blocks, and in Section 4, we discuss representative sets for acyclicity. In Section 5, we prove Theorems 1.1 and 1.3, respectively. Section 6 shows that if  $\mathcal{P}$  contains the cycle graph on  $d$  vertices, then both problems are not solvable in time  $2^{o(w \log w)} n^{\mathcal{O}(1)}$  on graphs of treewidth at most  $w$ , unless the ETH fails. In Section 7, we further show that if  $d$  is not fixed and  $\mathcal{P}$  contains all chordal graphs, then BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION is  $W[1]$ -hard when parameterized by both  $k$  and  $w$ .

## 2 Preliminaries

Let  $G$  be a graph. We denote the vertex set and edge set of  $G$  by  $V(G)$  and  $E(G)$ , respectively. For a vertex  $v$  in  $G$ , the *deletion* of  $v$  in  $G$  is the graph obtained by removing  $v$  and its incident edges, and is denoted  $G - v$ . For  $X \subseteq V(G)$ , we denote by  $G - X$  the deletion of every  $x \in X$ . For a vertex  $v$  in  $G$ , we denote by  $N_G(v)$  the set of neighbors of  $v$  in  $G$ , and  $N_G[v] := N_G(v) \cup \{v\}$ . For  $X \subseteq V(G)$ , we let  $N_G(X) := (\bigcup_{v \in X} N_G(v)) \setminus X$ . For two graphs  $G_1$  and  $G_2$ ,  $G_1 \cup G_2$  is the graph with the vertex set  $V(G_1) \cup V(G_2)$  and the edge set  $E(G_1) \cup E(G_2)$ , and  $G_1 \cap G_2$  is the graph with the vertex set  $V(G_1) \cap V(G_2)$  and the edge set  $E(G_1) \cap E(G_2)$ .

A vertex  $v$  of  $G$  is a *cut vertex* if the deletion of  $v$  from  $G$  increases the number of connected components. We say  $G$  is *biconnected* if it is connected and has no cut vertices. Note that every connected graph on at most two vertices is biconnected. A *block* of  $G$  is a maximal biconnected subgraph of  $G$ . We say  $G$  is *2-connected* if it is biconnected and  $|V(G)| \geq 3$ .

An induced cycle of length at least four is called a *chordless cycle*. A graph is *chordal* if it has no chordless cycles. For a class of graphs  $\mathcal{P}$ , a graph is called a  $\mathcal{P}$ -*block graph* if each of its blocks is in  $\mathcal{P}$ . A class  $\mathcal{C}$  of graphs is *hereditary* if for every  $G \in \mathcal{C}$  and every induced subgraph  $H$  of  $G$ ,  $H \in \mathcal{C}$ . A class  $\mathcal{C}$  of graphs is *block-hereditary* if for every  $G \in \mathcal{C}$  and every biconnected induced subgraph  $H$  of  $G$ ,  $H \in \mathcal{C}$ . Note that a block-hereditary class is not necessarily hereditary. For instance, if  $\mathcal{C}$  consists of  $K_1$ ,  $K_2$ , and cycle graphs, then  $\mathcal{C}$  is block-hereditary but not hereditary.

For a positive integer  $d$ , let  $[d] := \{1, \dots, d\}$ , and for two integers  $d_1, d_2$  with  $d_1 \leq d_2$ , let  $[d_1, d_2] := \{d_1, d_1 + 1, \dots, d_2\}$ . For a function  $f : X \rightarrow Y$  and  $X' \subseteq X$ , the function  $f' : X' \rightarrow Y$  where  $f'(x) = f(x)$  for all  $x \in X'$  is called the *restriction* of  $f$  on  $X'$ , and is denoted  $f|_{X'}$ . For such a pair of functions  $f$  and  $f'$ , we also say that  $f$  *extends*  $f'$  to the set  $X$ .

### 2.1 Block $d$ -labeling

A *block  $d$ -labeling* of a graph  $G$  is a function  $L : V(G) \rightarrow [d]$  such that for each block  $B$  of  $G$ ,  $L|_{V(B)}$  is an injection. If  $G$  is equipped with a block  $d$ -labeling  $L$ , then it is called a *block  $d$ -labeled graph*, and we call  $L(v)$  the *label* of  $v$ . Two block  $d$ -labeled graphs  $G$  and  $H$  are *label-isomorphic* if there is a graph isomorphism from  $G$  to  $H$  that is label preserving. For biconnected block  $d$ -labeled graphs  $G$  and  $H$ , we say  $H$  is *partially label-isomorphic* to  $G$  if  $H$  is label-isomorphic to the subgraph of  $G$  induced by the vertices with labels in  $H$ . Where there is no ambiguity, a block  $d$ -labeled graph will simply be called a  $d$ -labeled graph.

### 2.2 Treewidth

A *tree decomposition* of a graph  $G$  is a pair  $(T, \mathcal{B})$  consisting of a tree  $T$  and a family  $\mathcal{B} = \{B_t\}_{t \in V(T)}$  of sets  $B_t \subseteq V(G)$ , called *bags*, satisfying the following three conditions:

1.  $V(G) = \bigcup_{t \in V(T)} B_t$ ,
2. for every edge  $uv$  of  $G$ , there exists a node  $t$  of  $T$  such that  $u, v \in B_t$ , and
3. for  $t_1, t_2, t_3 \in V(T)$ ,  $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$  whenever  $t_2$  is on the path from  $t_1$  to  $t_3$  in  $T$ .

The *width* of a tree decomposition  $(T, \mathcal{B})$  is  $\max\{|B_t| - 1 : t \in V(T)\}$ . The *treewidth* of  $G$  is the minimum width over all tree decompositions of  $G$ . A *path decomposition* is a tree decomposition

$(P, \mathcal{B})$  where  $P$  is a path. The *pathwidth* of  $G$  is the minimum width over all path decompositions of  $G$ . We denote a path decomposition  $(P, \mathcal{B})$  as  $(B_{v_1}, \dots, B_{v_t})$ , where  $P$  is a path  $v_1 v_2 \dots v_t$ .

To design a dynamic programming algorithm, we use a convenient form of a tree decomposition known as a nice tree decomposition. A tree  $T$  is said to be *rooted* if it has a specified node called the *root*. Let  $T$  be a rooted tree with root node  $r$ . A node  $t$  of  $T$  is called a *leaf* node if it has degree one and it is not the root. For two nodes  $t_1$  and  $t_2$  of  $T$ ,  $t_1$  is a *descendant* of  $t_2$  if the unique path from  $t_1$  to  $r$  contains  $t_2$ . If a node  $t_1$  is a descendant of a node  $t_2$  and  $t_1 t_2 \in E(T)$ , then  $t_1$  is called a *child* of  $t_2$ .

A tree decomposition  $(T, \mathcal{B} = \{B_t\}_{t \in V(T)})$  is a *nice tree decomposition* with root node  $r \in V(T)$  if  $T$  is a rooted tree with root node  $r$ , and every node  $t$  of  $T$  is one of the following:

1. a *leaf node*:  $t$  is a leaf of  $T$  and  $B_t = \emptyset$ ;
2. a *introduce node*:  $t$  has exactly one child  $t'$  and  $B_t = B_{t'} \cup \{v\}$  for some  $v \in V(G) \setminus B_{t'}$ ;
3. a *forget node*:  $t$  has exactly one child  $t'$  and  $B_t = B_{t'} \setminus \{v\}$  for some  $v \in B_{t'}$ ; or
4. a *join node*:  $t$  has exactly two children  $t_1$  and  $t_2$ , and  $B_t = B_{t_1} = B_{t_2}$ .

**Theorem 2.1** (Bodlaender et al. [2]). *Given an  $n$ -vertex graph  $G$  and a positive integer  $k$ , one can either output a tree decomposition of  $G$  with width at most  $5k + 4$ , or correctly answer that the treewidth of  $G$  is larger than  $k$ , in time  $2^{\mathcal{O}(k)} n$ .*

**Lemma 2.2** (folklore; see Lemma 7.4 in [5]). *Given a tree decomposition of an  $n$ -vertex graph  $G$  of width  $w$ , one can construct a nice tree decomposition  $(T, \mathcal{B})$  of width  $w$  with  $|V(T)| = \mathcal{O}(wn)$  in time  $\mathcal{O}(k^2 \cdot \max(|V(T)|, |V(G)|))$ .*

### 2.3 Boundaried graphs

For a graph  $G$  and  $S \subseteq V(G)$ , the pair  $(G, S)$  is called a *boundaried graph*. When  $G$  is a  $d$ -labeled graph, we simply say that  $(G, S)$  is a  $d$ -labeled graph. Two  $d$ -labeled graphs  $(G, S)$  and  $(H, S)$  are said to be *compatible* if  $V(G - S) \cap V(H - S) = \emptyset$ ,  $G[S] = H[S]$ , and  $G$  and  $H$  have the same labels on  $S$ . For two compatible  $d$ -labeled graphs  $(G, S)$  and  $(H, S)$ , the *sum* of two graphs is the graph obtained from the disjoint union of  $G$  and  $H$  by identifying each vertex in  $S$  and removing an edge if multiple edges appear, and is denoted by  $(G, S) \oplus (H, S)$ . We also denote by  $L_G \oplus L_H$  the function from the vertex set of  $(G, S) \oplus (H, S)$  to  $[d]$  where for  $v \in V(G) \cup V(H)$ ,  $(L_G \oplus L_H)(v) = L_G(v)$  if  $v \in V(G)$  and  $(L_G \oplus L_H)(v) = L_H(v)$  otherwise. Notice that  $L_G \oplus L_H$  is not necessarily a block  $d$ -labeling of  $G \oplus H$ . For two unlabeled boundaried graphs, we define the sum in the same way, but ignoring the label condition.

A block of a graph is *non-trivial* if it has at least two vertices. For a boundaried graph  $(G, S)$ , a block  $B$  of  $G$  is called an  *$S$ -block* if it contains an edge of  $G[S]$ . Note that every non-trivial block of  $G[S]$  is contained in a unique  $S$ -block of  $G$  because two distinct blocks share at most one vertex. Let  $(G, S)$  be a boundaried graph. We define  $\mathbf{Aux}(G, S)$  as the bipartite boundaried graph with bipartition  $(\mathcal{C}_1, \mathcal{C}_2)$  and boundary  $\mathcal{C}_2$  such that

1.  $\mathcal{C}_1$  is the set of components of  $G$ , and  $\mathcal{C}_2$  is the set of components of  $G[S]$ , and
2. for  $C_1 \in \mathcal{C}_1$  and  $C_2 \in \mathcal{C}_2$ ,  $C_1 C_2 \in E(\mathbf{Aux}(G, S))$  if and only if  $C_2$  is contained in  $C_1$ .

We remark that when  $(G, S)$  and  $(H, S)$  are two compatible  $d$ -labeled graphs,  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  is well-defined, as  $G$  and  $H$  have the same set of components on  $S$ .

### 3 Lemmas about $S$ -blocks

In this section, we present several lemmas regarding  $S$ -blocks. For a biconnected  $d$ -labeled graph  $Q$ , we say a  $d$ -labeled graph  $(G, S)$  is *block-wise partially label-isomorphic to  $Q$*  if every  $S$ -block  $B$  of  $G$  is partially label-isomorphic to  $Q$ . For two compatible  $d$ -labeled graphs  $(G, S)$  and  $(H, S)$  with labelings  $L_G$  and  $L_H$  respectively, we say  $(G, S)$  and  $(H, S)$  are *block-wise  $Q$ -compatible* if

1.  $(G, S)$  and  $(H, S)$  are block-wise partially label-isomorphic to  $Q$ ; and
2. for every non-trivial block  $B$  of  $G[S]$ , letting  $B_1$  and  $B_2$  be the  $S$ -blocks of  $G$  and  $H$  that contain  $B$ , respectively,  $L_G(N_{B_1}(V(B)) \setminus S) \cap L_H(N_{B_2}(V(B)) \setminus S) = \emptyset$ , and, for  $\ell_1 \in L_G(N_{B_1}(V(B)) \setminus S)$  and  $\ell_2 \in L_H(N_{B_2}(V(B)) \setminus S)$ , the vertices in  $Q$  with labels  $\ell_1$  and  $\ell_2$  are not adjacent.

At first, we describe sufficient conditions for when, given a chordal labeled graph  $Q$ , the sum of two given labeled graphs  $(G, S)$  and  $(H, S)$ , each partially label-isomorphic to  $Q$ , is again partially label-isomorphic to  $Q$ . For this argument, we need the assumption that  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles.

**Proposition 3.1.** *Let  $Q$  be a biconnected  $d$ -labeled chordal graph. Let  $(G, S)$  and  $(H, S)$  be two block-wise  $Q$ -compatible  $d$ -labeled graphs such that  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. Then  $(G, S) \oplus (H, S)$  is block-wise partially label-isomorphic to  $Q$ .*

The following lemma is an essential property of chordal graphs:

**Lemma 3.2.** *Let  $F$  be a connected graph and let  $Q$  be a connected chordal graph. Let  $\mu : V(F) \rightarrow V(Q)$  be a function such that for every induced path  $p_1 \cdots p_m$  in  $F$  of length at most two,  $\mu(p_1), \dots, \mu(p_m)$  are pairwise distinct and  $\mu(p_1) \cdots \mu(p_m)$  is an induced path of  $Q$ . Then  $\mu$  is an injection and preserves the adjacency relation.*

*Proof.* We first show that  $\mu$  is an injection.

**Claim 3.2.1.**  *$F$  has no two vertices  $v$  and  $w$  with  $\mu(v) = \mu(w)$ .*

*Proof.* Suppose  $F$  has two distinct vertices  $v$  and  $w$  with  $\mu(v) = \mu(w)$ . Let us choose such vertices  $v$  and  $w$  with minimum distance in  $F$ , and let  $P = p_1 p_2 \cdots p_x$  be a shortest path from  $v = p_1$  to  $w = p_x$  in  $F$ . Note that  $P$  is an induced path, and by assumption,  $x \geq 4$  and  $\mu(p_1)\mu(p_2)\mu(p_3)$  is an induced path in  $Q$ . This further implies that  $\mu(p_4) \neq \mu(p_i)$  for  $i \in \{1, 2, 3\}$ . Thus, we have that  $x \geq 5$ .

Let  $y \in \{4, \dots, x-1\}$  be the smallest integer such that  $\mu(p_y)$  is adjacent to one of  $\mu(p_1), \dots, \mu(p_{y-3})$ . Such an integer exists as  $\mu(p_1) = \mu(p_x)$ , so  $\mu(p_{x-1})$  is adjacent to  $\mu(p_1)$ , and  $\mu(p_i)\mu(p_{i+1})\mu(p_{i+2})$  is an induced path for each  $1 \leq i \leq x-2$ . Let  $\mu(p_z)$  be a neighbor of  $\mu(p_y)$  with  $z \in \{1, 2, \dots, y-3\}$  and maximum  $z$ . Then  $\mu(p_z)\mu(p_{z+1}) \cdots \mu(p_y)\mu(p_z)$  is an induced cycle of length at least 4, which contradicts the assumption that  $Q$  is chordal.  $\diamond$

Now, we show that  $\mu$  preserves the adjacency relation.

**Claim 3.2.2.** *For each  $v, w \in V(F)$ ,  $vw \in E(F)$  if and only if  $\mu(v)\mu(w) \in E(Q)$ .*

*Proof.* Suppose there are two vertices  $v$  and  $w$  in  $F$  such that the adjacency between  $v$  and  $w$  in  $F$  is different from the adjacency between  $\mu(v)$  and  $\mu(w)$  in  $Q$ . When  $vw \in E(F)$ ,  $\mu(v)$  is adjacent to  $\mu(w)$  in  $Q$  by assumption. Thus,  $vw \notin E(F)$  and  $\mu(v)\mu(w) \in E(Q)$ . We choose such vertices  $v$  and  $w$  with minimum distance in  $F$ . Let  $P = p_1p_2 \cdots p_x$  be a shortest path from  $v = p_1$  to  $w = p_x$  in  $F$ . Observe that  $x \geq 4$ . By the minimality of the distance, each of  $\mu(p_1)\mu(p_2) \cdots \mu(p_{x-1})$  and  $\mu(p_2)\mu(p_3) \cdots \mu(p_x)$  is an induced path in  $Q$ . Therefore,  $\mu(p_1)\mu(p_2) \cdots \mu(p_x)\mu(p_1)$  is an induced cycle of length at least four in  $Q$ , contradicting the assumption that  $Q$  is chordal.  $\diamond$

This completes the proof.  $\square$

We need two auxiliary lemmas to prove Proposition 3.1.

**Lemma 3.3.** *Let  $(G, S)$  and  $(H, S)$  be two compatible  $d$ -labeled graphs such that  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. If  $F$  is an  $S$ -block of  $(G, S) \oplus (H, S)$  and  $uv$  is an edge in  $F$ , then  $uv$  is contained in some  $S$ -block of  $G$  or  $H$ .*

*Proof.* Let  $uv \in E(F)$ . We may assume that one of  $u$  and  $v$  is not contained in  $S$ . Without loss of generality, let us assume  $v \in V(G) \setminus S$ . The same proof holds when one of  $u$  and  $v$  is in  $V(H) \setminus S$ .

Let  $C_v$  be the component of  $G$  containing  $v$  and let  $B$  be the block of  $G$  containing  $u$  and  $v$ . Suppose towards a contradiction that  $B$  is not an  $S$ -block. As  $F$  is an  $S$ -block of  $(G, S) \oplus (H, S)$ ,  $F$  contains an edge of  $G[S]$ , say  $e$ . Let  $D'$  be the component of  $G[S]$  containing  $e$ , and let  $D$  be the component of  $G[S]$  that is on the path from  $C_v$  to  $D'$  and is adjacent to  $C_v$  in  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ .

Now, for each cut vertex  $w$  of  $G$  contained in  $B$ , let  $H_w$  be the subgraph of  $G$  induced by the union of  $w$  and all components of  $C_v - B$  containing a neighbor of  $w$ . One can observe that for distinct cut vertices  $w_1$  and  $w_2$  of  $G$  on  $B$ ,  $H_{w_1}$  and  $H_{w_2}$  cannot have vertices from the same component of  $G[S]$ . So, there is a unique cut vertex  $w$  of  $G$  on  $B$ , where  $H_w$  contains a vertex of  $D$ . This implies that  $w$  separates  $D$  from  $\{u, v\}$  in  $G$ , and furthermore, since  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles,  $w$  separates  $e$  from  $\{u, v\}$  in  $(G, S) \oplus (H, S)$ . This contradicts the assumption that  $F$  is 2-connected. We conclude that  $B$  is an  $S$ -block.  $\square$

**Lemma 3.4.** *Let  $(G, S)$  and  $(H, S)$  be two compatible  $d$ -labeled graphs such that each  $S$ -block of  $G$  or  $H$  is chordal, and  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. If  $F$  is an  $S$ -block of  $(G, S) \oplus (H, S)$  and  $uvw$  is an induced path in  $F$  such that  $u$  and  $w$  are not contained in the same  $S$ -block of  $G$  or  $H$ , then*

1.  $v \in S$ , and
2. there is an induced path  $q_1q_2 \cdots q_\ell$  from  $u = q_1$  to  $w = q_\ell$  in  $F - v$  such that each  $q_i$  is a neighbor of  $v$ .

*Proof.* Since  $F$  contains at least 3 vertices,  $F$  is 2-connected. Let  $C$  be the component of  $G$  containing  $v$ .

(1) We verify that  $v \in S$ . Suppose  $v \notin S$ , and without loss of generality we assume  $v \in V(G) \setminus S$ . By Lemma 3.3, each of  $uv$  and  $vw$  is contained in some  $S$ -block of  $G$  or  $H$ . But since  $u$  and  $w$  are not contained in the same block,  $v$  is a cut vertex of  $G$ . Let  $H_1$  be the subgraph of  $G$  induced by the union of  $v$  and the component of  $C - v$  containing  $u$ , and let  $H_2$  be the subgraph of  $G$  induced



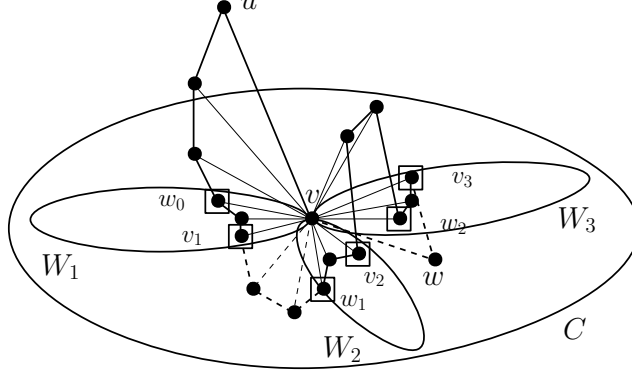


Figure 1: The required path from  $u$  to  $w$  described in Lemma 3.4. Dashed edges denote edges incident with vertices in  $H - S$ .

by the union of  $v$  and the component of  $C - v$  containing  $w$ . Then  $H_1$  and  $H_2$  do not contain vertices from the same component of  $G[S]$ . This implies that  $v$  separates  $u$  and  $w$  in  $G$ , and since  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles,  $v$  separates  $u$  and  $w$  in  $(G, S) \oplus (H, S)$ . This contradicts the assumption that  $F$  is 2-connected. Therefore, we have  $v \in S$ .

(2) Let  $D$  be the connected component of  $G[S]$  containing  $v$ , and let  $U_1, \dots, U_p$  be the set of all induced subgraphs of  $D$  consisting of  $v$  and one of the connected components of  $D - v$ . As  $v \in V(D)$ , we can observe that for each  $z \in \{u, w\}$ , either  $z \in V(G) \setminus S$  or  $z \in V(H) \setminus S$  or  $z \in V(D) \setminus \{v\}$ .

We verify that for each  $z \in \{u, w\}$ , there is a path from  $z$  to  $D - v$  in  $G - v$  or  $H - v$ . If  $z \in V(D) \setminus \{v\}$ , then this is clear. By symmetry, we may assume  $z \in V(G) \setminus S$ . We claim that there is a path from  $z$  to  $D - v$  in  $G - v$ . Suppose there is no path from  $z$  to  $D - v$  in  $G - v$ . Thus,  $v$  is a cut vertex of  $G$  separating  $z$  from  $D - v$ . Let  $H_3$  be the subgraph of  $G$  induced by the union of  $v$  and the component of  $C - v$  containing  $z$ . Since  $u$  and  $w$  are not contained in the same block,  $H_3$  does not contain the other vertex in  $\{u, w\} \setminus \{z\}$ . Also, since  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles,  $u$  separates  $v$  and  $w$  in  $(G, S) \oplus (H, S)$ . This contradicts the assumption that  $F$  is 2-connected. We conclude that there is a path from  $z$  to  $D - v$  in  $G - v$ .

Let  $W_1 - W_2 - \dots - W_m$  be the shortest sequence of  $\{U_1, \dots, U_p\}$  such that

- there is a path from  $u$  to  $W_1$  in  $G - v$  or  $H - v$ ,
- there is a path from  $w$  to  $W_m$  in  $G - v$  or  $H - v$ , and
- if  $m \geq 2$ , then for each  $i \in \{1, \dots, m - 1\}$ , there is a path from  $W_i - v$  to  $W_{i+1} - v$  in  $G - v$  or  $H - v$ .

Such a sequence always exists as  $F - v$  is connected and  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. Now, we construct the required path. See Figure 1 for an illustration.

Let  $P_0 = z_1 z_2 \dots z_\ell$  be a shortest path from  $u = z_1$  to  $w_0 = z_\ell \in V(W_1) \setminus \{v\}$  in  $G - v$  or  $H - v$  such that the distance from  $w_0$  to  $v$  in  $W_1$  is minimum. Let  $R$  be a shortest path from  $z_\ell$  to  $v$  in  $W_1$ . As  $G[V(P_0) \cup V(R)]$  is 2-connected, it is contained in an  $S$ -block, and by assumption, it is chordal. We claim that every vertex in  $P_0$  is a neighbor of  $v$ . Suppose there exists  $i \in \{2, \dots, \ell - 1\}$  such that  $z_i$  is not adjacent to  $v$ . By the distance condition, there are no edges between  $\{z_1, \dots, z_{i-1}\}$  and  $\{z_{i+1}, \dots, z_\ell\} \cup (V(W_1) \setminus \{v\})$ . Merging a shortest path from  $z_i$  to  $v$  in  $G[\{z_1, \dots, z_i\} \cup \{v\}]$  and a

shortest path from  $z_i$  to  $v$  in  $G[\{z_i, \dots, z_\ell\} \cup V(R)]$ , one can find a chordless cycle in  $G[V(P_0) \cup V(R)]$ ; a contradiction. Therefore, every vertex in  $V(P_0) \setminus \{z_\ell\}$  is a neighbor of  $v$ . Finally, by the assumption that the distance from  $w_0$  to  $v$  in  $W_1$  is minimum,  $w_0$  is a neighbor of  $v$ . Also, we can observe that every vertex in  $P_0$  is in  $F$ .

Let  $P_m$  be a shortest path from  $w$  to  $v_m \in V(W_m) \setminus \{v\}$  such that the distance from  $w_0$  to  $v$  in  $W_1$  is minimum. Also, for each  $i \in \{1, \dots, m-1\}$ , let  $P_i$  be the shortest path from  $v_i \in V(W_i) \setminus \{v\}$  to  $w_i \in V(W_{i+1}) \setminus \{v\}$  in  $G-v$  or  $H-v$  such that the distance from  $v_i$  to  $v$  in  $W_i$  and the distance from  $w_i$  to  $v$  in  $W_{i+1}$  are minimum. Lastly, for each  $i \in \{1, \dots, m\}$ , let  $Q_i$  be a shortest path from  $w_{i-1}$  to  $v_i$  in  $W_i - v$ . Similar to  $P_0$ , we can prove that every vertex of  $Q_1 \cup P_1 \cup \dots \cup Q_m \cup P_m$  is a neighbor of  $v$ , and is contained in  $F$ . Therefore, the shortest path from  $u$  to  $w$  in  $P_0 \cup Q_1 \cup P_1 \cup \dots \cup Q_m \cup P_m$  is the required path.  $\square$

*Proof of Proposition 3.1.* Let  $F$  be an  $S$ -block of  $(G, S) \oplus (H, S)$ . Let  $L_G$  and  $L_H$  be labelings of  $G$  and  $H$ , respectively, and let  $L := L_G \oplus L_H$ . We may assume  $|V(F)| \geq 3$  and thus  $F$  is 2-connected. By Lemma 3.3, every edge of  $F$  is contained in some  $S$ -block of  $G$  or  $H$ . This implies that for every edge  $uv$  of  $F$ , we have  $L(u) \neq L(v)$  and the vertices with labels  $L(u)$  and  $L(v)$  are adjacent in  $Q$ . Moreover, since  $(G, S)$  and  $(H, S)$  are block-wise partially label-isomorphic to  $Q$ , we have  $L(V(F)) \subseteq L_Q(V(Q))$ . Let  $\mu : V(F) \rightarrow V(Q)$  such that for each  $v \in V(F)$ ,  $L(v) = L_Q(\mu(v))$ .

To apply Lemma 3.2, it is sufficient to prove the following. Notice that we do not know yet whether  $F$  is chordal or not.

**Claim 3.4.1.** *If  $uvw$  is an induced path in  $F$ , then  $L(u) \neq L(w)$  and  $\mu(u)\mu(v)\mu(w)$  is an induced path in  $Q$ .*

*Proof.* Since  $(G, S)$  and  $(H, S)$  are block-wise partially label-isomorphic to  $Q$ , if all of  $u, v, w$  are contained in an  $S$ -block of  $G$  or  $H$ , then it follows from the given condition. We may assume  $u$  and  $w$  are not contained in the same  $S$ -block of  $G$  or  $H$ . Then by Lemma 3.4,  $v \in S$ , and there is an induced path  $q_1 q_2 \dots q_\ell$  from  $u = q_1$  to  $w = q_\ell$  in  $F - v$  such that each  $q_i$  is a neighbor of  $v$ .

We show that for each  $i \in \{1, \dots, \ell-2\}$ ,  $L(q_i), L(q_{i+1}), L(q_{i+2})$  are pairwise distinct, and  $\mu(q_i)\mu(q_{i+1})\mu(q_{i+2})$  is an induced path of  $Q$ . If all of  $q_i, q_{i+1}, q_{i+2}$  are contained in  $G$  or  $H$ , then they are contained in the same  $S$ -block as  $v$ , and the claim follows. We may assume  $q_i$  and  $q_{i+2}$  are contained in distinct graphs of  $G - S$  and  $H - S$ . Then the  $S$ -block containing  $q_i, q_{i+1}, v$  and the  $S$ -block containing  $q_{i+1}, q_{i+2}, v$  share the edge  $q_{i+1}v$ . Since  $(G, S)$  and  $(H, S)$  are block-wise  $Q$ -compatible,  $L(q_i) \neq L(q_{i+2})$  and  $\mu(q_i)$  is not adjacent to  $\mu(q_{i+2})$  in  $Q$ .

Similarly, we verify that  $\mu(q_1)\mu(q_2) \dots \mu(q_\ell)$  is an induced path of  $Q$ . Suppose this is false, and choose  $i_1, i_2 \in \{1, 2, \dots, \ell\}$  with  $i_2 - i_1 > 1$  and minimum  $i_2 - i_1$  such that  $\mu(q_{i_1})$  is adjacent to  $\mu(q_{i_2})$  in  $Q$ . By minimality,  $\mu(q_{i_1}) \dots \mu(q_{i_2-1})$  and  $\mu(q_{i_1+1}) \dots \mu(q_{i_2})$  are induced paths and have length at least 2. Thus  $\mu(q_{i_1}) \dots \mu(q_{i_2})$  is an induced cycle of length at least 4, contradicting the assumption that  $Q$  is chordal. Therefore,  $\mu(q_1)\mu(q_2) \dots \mu(q_\ell)$  is an induced path of  $Q$ , and, in particular,  $L(u) \neq L(w)$  and  $\mu(u)$  and  $\mu(w)$  are not adjacent in  $Q$ , as required.  $\diamond$

By Claim 3.4.1 and Lemma 3.2, we conclude that  $F$  is partially label-isomorphic to  $Q$ .  $\square$

Later, we will consider some information on non-trivial blocks of  $G[S]$ , where if two blocks are contained in the same  $S$ -block of  $G$  or  $H$ , then they have the same information. In Lemma 3.5, we analyze when this property is preserved after taking the sum of  $(G, S)$  and  $(H, S)$ .

**Lemma 3.5.** *Let  $A$  be a set, let  $(G, S)$  and  $(H, S)$  be two compatible  $d$ -labeled graphs, let  $\mathcal{B}$  be the set of non-trivial blocks in  $G[S]$ , and let  $g : \mathcal{B} \rightarrow A$  be a function such that*

- *each  $S$ -block of  $G$  or  $H$  is chordal,*
- *$\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles, and*
- *for every  $B_1, B_2 \in \mathcal{B}$  where  $B_1$  and  $B_2$  are contained in an  $S$ -block of  $G$  or  $H$ ,  $g(B_1) = g(B_2)$ .*

*If  $F$  is an  $S$ -block of  $(G, S) \oplus (H, S)$  and  $B_1, B_2 \in \mathcal{B}$  where  $V(B_1), V(B_2) \subseteq V(F)$ , then  $g(B_1) = g(B_2)$ .*

*Proof.* By Lemma 3.3, every edge of  $F$  is contained in an  $S$ -block of  $G$  or  $H$ . We define a function  $g' : E(F) \rightarrow A$  such that for each  $vw \in E(F)$ ,  $g'(vw) = g(B)$  where  $B \in \mathcal{B}$  and  $B$  is contained in the  $S$ -block of  $G$  or  $H$  containing  $v$  and  $w$ . To prove this, it is sufficient to show that  $g'(e) = g'(f)$  for all  $e, f \in E(F)$ . Suppose towards a contradiction that there are  $e, f \in E(F)$  such that  $e$  and  $f$  share a vertex and  $g'(e) \neq g'(f)$ . Let  $e = uv$  and  $f = vw$ . Then  $u, v, w$  are not contained in the same  $S$ -block of  $G$  or  $H$  as  $g'(e) \neq g'(f)$ . Thus by Lemma 3.4,  $v \in S$ , and there is an induced path  $q_1 q_2 \cdots q_\ell$  from  $u = q_1$  to  $w = q_\ell$  in  $F - v$  such that each  $q_i$  is a neighbor of  $v$ .

As  $q_1, q_2, v$  are contained in the same  $S$ -block of  $G$  or  $H$ , we observe that  $g'(q_1 q_2) = g'(q_1 v) = g'(uv)$ . Similarly, we have  $g'(q_{\ell-1} q_\ell) = g'(q_\ell w) = g'(vw)$ . We claim that for each  $i \in \{1, \dots, \ell - 2\}$ ,  $g'(q_i q_{i+1}) = g'(q_{i+1} q_{i+2})$ . Let  $i \in \{1, \dots, \ell - 2\}$ . If all of  $q_i, q_{i+1}, q_{i+2}$  are contained in  $G$  or  $H$ , then they are contained in the same  $S$ -block as  $v$ , and the claim follows. We may assume  $q_i$  and  $q_{i+2}$  are contained in distinct graphs of  $G - S$  and  $H - S$ . Then the  $S$ -block containing  $q_i, q_{i+1}, v$  and the  $S$ -block containing  $q_{i+1}, q_{i+2}, v$  share the edge  $q_{i+1} v$ , and we have  $g'(q_i q_{i+1}) = g'(q_{i+1} v) = g'(q_{i+1} q_{i+2})$ . Therefore,  $g'(uv) = g'(q_1 q_2) = g'(q_{\ell-1} q_\ell) = g'(vw)$ , which is a contradiction. We conclude that  $g'(e) = g'(f)$  for all  $e, f \in E(F)$ , as required.  $\square$

We also need the following lemma.

**Lemma 3.6.** *Let  $(G, S)$  and  $(H, S)$  be two compatible  $d$ -labeled graphs such that  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. If  $F$  is an  $S$ -block of  $(G, S) \oplus (H, S)$ , then  $\mathbf{Aux}(F \cap G, S \cap V(F)) \oplus \mathbf{Aux}(F \cap H, S \cap V(F))$  has no cycles.*

*Proof.* Let  $S_F := S \cap V(F)$ . Suppose towards a contradiction that  $\mathbf{Aux}(F \cap G, S_F) \oplus \mathbf{Aux}(F \cap H, S_F)$  has a cycle  $C_1 - F_1 - \cdots - C_m - F_m - C_1$ , where  $C_1, \dots, C_m$  are components of  $F[S_F]$  and  $m \geq 2$ .

First assume that there are two distinct components  $C_i$  and  $C_j$  of  $F[S_F]$  contained in the same component of  $G[S]$ . We choose such components  $C_i, C_j \in \{C_1, \dots, C_m\}$  such that the distance between  $C_i$  and  $C_j$  in the cycle  $C_1 - F_1 - \cdots - C_m - F_m - C_1$  is minimum. By relabeling if necessary, we may assume  $i < j$  and in the sequence  $C_i, C_{i+1}, \dots, C_j$ , there are no two components contained in the same component of  $G[S]$  except the pair  $(C_i, C_j)$ .

Since  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles, we observe that all of  $C_i, F_i, C_{i+1}, F_{i+1}, \dots, C_j$  are contained in the same component of  $G$  or  $H$ . As  $C_i$  and  $C_j$  are contained in the same component of  $G[S]$ , there is a cycle of  $G$  or  $H$  passing through all of  $C_i, F_i, C_{i+1}, F_{i+1}, \dots, C_j$  and the component of  $G[S]$  containing  $C_i$  and  $C_j$ . This implies that  $C_i, F_i, C_{i+1}, F_{i+1}, \dots, C_j$  are contained in the same component of  $F \cap G$  or  $F \cap H$ . This contradicts the assumption that  $C_i$  and  $C_j$  are distinct components of  $F[S_F]$ . We conclude that there are no two distinct components  $C_i$  and  $C_j$  contained in the same component of  $G[S]$ .



$x$  to  $y$  along the other part of the cycle  $v_1P_1 - Q_1 - P_2 - Q_2 - \dots - P_n - Q_nv_1$ , then  $P \cup P'$  is a chordless cycle. This proves the claim.

(2  $\Rightarrow$  1). Suppose, towards a contradiction, that  $(G, S) \oplus (H, S)$  contains a chordless cycle  $C$ . Since  $G$  and  $H$  are chordal,  $C$  should contain a vertex of  $G - S$  and a vertex of  $H - S$ . By assumption, we know that every  $S$ -block of  $(G, S) \oplus (H, S)$  is chordal. Thus,  $C$  can contain at most one vertex from each  $S$ -block of  $(G, S) \oplus (H, S)$ . Furthermore, we can observe that  $|V(C) \cap V(F)| \leq 1$  for every connected component  $F$  of  $G[S]$ , otherwise one of  $S$ -blocks of  $(G, S) \oplus (H, S)$  should contain all vertices of  $C$ , contradicting the fact that every  $S$ -block is chordal.

Let  $C_1 - C_2 - \dots - C_n - C_1$  be the sequence of connected components of  $G[S]$  such that

1. for each  $v \in V(C) \cap V(C_i)$ , one neighbor of  $v$  in  $C$  is contained in  $G - S$  and the other is contained in  $H - S$ , and
2.  $C$  passes through the connected components of  $G[S]$  in this order.

As  $C$  contains at least one vertex of  $G - S$  and one vertex of  $H - S$ , such a sequence exists, and  $n \geq 2$ . Without loss of generality, we may assume that the internal vertices in the path from  $C_1$  to  $C_2$  (corresponding to the first part of the sequence) are contained in  $G$ . Then, the internal vertices in the path from  $C_2$  to  $C_3$  are contained in  $H$ , and we use parts of  $G - S$  and  $H - S$  alternately. For each  $i$ , pick  $A_i \in V(\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)) \setminus C$  corresponding to a connected component of  $G$  or  $H$  containing the internal vertices of the path from  $C_i$  to  $C_{i+1}$ . Then  $C_1 - A_1 - C_2 - A_2 - \dots - C_n - A_n - C_1$  contains a cycle of  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$ .  $\square$

## 4 Representative sets for acyclicity

In our algorithm, we need to store auxiliary graphs  $\mathbf{Aux}(G, S)$  for boundaried graphs  $(G, S)$ . Instead of working with  $\mathbf{Aux}(G, S)$ , we instead work with the partition of the set  $\mathcal{C}$  of connected components of  $G[S]$ , where  $C_1, C_2 \in \mathcal{C}$  are in the same part if and only if they are contained in the same component of  $G$ . This formulation has the advantage that it is convenient for representative-set techniques.

For a set  $S$  and a family  $\mathcal{X}$  of subsets of  $S$ , let  $\mathbf{Inc}(S, \mathcal{X})$  be the bipartite graph on the bipartition  $(S, \mathcal{X})$  where for  $v \in S$  and  $X \in \mathcal{X}$ ,  $v$  and  $X$  are adjacent in  $\mathbf{Inc}(S, \mathcal{X})$  if and only if  $v \in X$ . Let  $S$  be a set, and let  $\mathcal{A}$  be a set of partitions of  $S$ . A subset  $\mathcal{A}'$  of  $\mathcal{A}$  is called a *representative set* if

- for every  $\mathcal{X}_1 \in \mathcal{A}$  and every partition  $\mathcal{Y}$  of  $S$  where  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{Y})$  has no cycles, there exists a partition  $\mathcal{X}_2 \in \mathcal{A}'$  such that  $\mathbf{Inc}(S, \mathcal{X}_2 \cup \mathcal{Y})$  has no cycles.

Computing a representative set for a family of partitions is an essential part of our algorithm. To apply the ideas in [1], it is necessary to translate our problem to finding a pair of partitions  $\mathcal{X}_1, \mathcal{X}_2$  where  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{X}_2)$  is connected. We argue by restricting the size of partitions in  $\mathcal{A}$ .

For partitions  $\mathcal{X}_1$  and  $\mathcal{X}_2$  of a set  $S$ ,  $\mathcal{X}_1$  is a *coarsening* of  $\mathcal{X}_2$  if every two elements in the same part of  $\mathcal{X}_2$  are in the same part of  $\mathcal{X}_1$ , and we denote by  $\mathcal{X}_1 \uplus \mathcal{X}_2$  the common coarsening of  $\mathcal{X}_1$  and  $\mathcal{X}_2$  with the maximum number of parts. For instance, if  $\mathcal{X}_1 = \{\{1\}, \{2, 3\}, \{4\}\}$  and  $\mathcal{X}_2 = \{\{1, 2\}, \{3\}, \{4\}\}$ , then both  $\{\{1, 2, 3\}, \{4\}\}$  and  $\{\{1, 2, 3, 4\}\}$  are common coarsenings of  $\mathcal{X}_1$  and  $\mathcal{X}_2$ , and  $\mathcal{X}_1 \uplus \mathcal{X}_2 = \{\{1, 2, 3\}, \{4\}\}$ .

**Lemma 4.1.** *Let  $S$  be a set and let  $\mathcal{X}_1, \mathcal{X}_2$  be two partitions of  $S$  such that  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{X}_2)$  is connected. Then  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{X}_2)$  has no cycles if and only if  $|\mathcal{X}_1| + |\mathcal{X}_2| = |S| + 1$ .*

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**Algorithm 1** REPARTITIONS( $S, \mathcal{A}$ )

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**Input:** A set  $S$  and a family  $\mathcal{A}$  of partitions of  $S$ .

**Output:** A representative set  $\mathcal{R}$  of  $\mathcal{A}$ .

- 1: We compute the family  $\mathcal{A}'$  of all 1-coarsenings of partitions in  $\mathcal{A}$ .
  - 2: For each  $1 \leq i \leq |S|$ , let  $\mathcal{A}_i := \{\mathcal{X} \in \mathcal{A}' : |\mathcal{X}| = i\}$  and let  $\mathcal{B}_i$  be the set of all partitions of  $S$  of size  $i$ .
  - 3: For each  $1 \leq i, j \leq |S|$  with  $i + j = |S| + 1$ , we obtain a set  $\mathcal{R}_i$  from  $\mathcal{A}_i$  with respect to  $\mathcal{B}_j$  using Theorem 4.2.
  - 4: We take the set  $\mathcal{R}$  from  $\bigcup_{1 \leq i \leq |S|} \mathcal{R}_i$  by taking the original partition before taking a 1-coarsening, and output  $\mathcal{R}$ .
- 

*Proof.* Let  $H := \mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{X}_2)$ . The result follows from the fact that  $|V(H)| = |S| + |\mathcal{X}_1| + |\mathcal{X}_2|$ ,  $|E(H)| = 2|S|$ , and a connected graph  $H$  has no cycles if and only if  $|E(H)| = |V(H)| - 1$ .  $\square$

For a set  $S$  and a partition  $\mathcal{X}$  of  $S$ , a partition  $\mathcal{Y}$  of  $S$  is called a 1-coarsening of  $\mathcal{X}$  if  $\mathcal{Y} = \mathcal{X} \setminus \{X_1, \dots, X_m\} \cup \{X_1 \cup \dots \cup X_m\}$  for some  $X_1, \dots, X_m \in \mathcal{X}$ . Notice that the partition  $\mathcal{X}$  itself is a 1-coarsening of  $\mathcal{X}$ . We will use the following observation. For two partitions  $\mathcal{X}_1, \mathcal{X}_2$  of a set  $S$ , the following are equivalent:

- $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{X}_2)$  has no cycles.
- There exists a 1-coarsening  $\mathcal{X}'_1$  of  $\mathcal{X}_1$  such that  $\mathbf{Inc}(S, \mathcal{X}'_1 \cup \mathcal{X}_2)$  is connected and has no cycles.

Such a 1-coarsening  $\mathcal{X}'_1$  can be obtained by taking one part of  $\mathcal{X}_1$  for each connected component of  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{X}_2)$  and unifying them into one part. Since the vertex corresponding to the new part of  $\mathcal{X}'_1$  would be a cut vertex of  $\mathbf{Inc}(S, \mathcal{X}'_1 \cup \mathcal{X}_2)$ , there will not be an additional cycle in  $\mathbf{Inc}(S, \mathcal{X}'_1 \cup \mathcal{X}_2)$  while it is connected.

We explicitly describe a necessary subroutine, Algorithm 1.

**Theorem 4.2** ([1]; See also Theorem 11.11 in [5]). *Given two families of partitions  $\mathcal{A}, \mathcal{B}$  of a set  $S$ , one can in time  $\mathcal{A}^{\mathcal{O}(1)} 2^{\mathcal{O}(|S|)}$  find a set  $\mathcal{A}' \subseteq \mathcal{A}$  of size at most  $2^{|S|-1}$  such that for every  $\mathcal{X}_1 \in \mathcal{A}$  and every  $\mathcal{Y} \in \mathcal{B}$  such that  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{Y})$  is connected, there exists  $\mathcal{X}_2 \in \mathcal{A}'$  such that  $\mathbf{Inc}(S, \mathcal{X}_2 \cup \mathcal{Y})$  is connected.*

**Proposition 4.3.** *Given a family  $\mathcal{A}$  of partitions of a set  $S$ , Algorithm 1 outputs a representative set of  $\mathcal{A}$  of size at most  $|S| \cdot 2^{|S|-1}$  in time  $\mathcal{A}^{\mathcal{O}(1)} 2^{\mathcal{O}(|S|)}$ .*

*Proof.* Let  $\mathcal{R}$  be the output of Algorithm 1. Clearly,  $\mathcal{R} \subseteq \mathcal{A}$ , because we take the original partitions of  $\bigcup_{1 \leq i \leq |S|} \mathcal{R}_i$  at the last step. Thus, it is sufficient to show that

- for every  $\mathcal{X}_1 \in \mathcal{A}$  and every partition  $\mathcal{Y}$  of  $S$  where  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{Y})$  has no cycles, there exists a partition  $\mathcal{X}_2 \in \mathcal{R}$  such that  $\mathbf{Inc}(S, \mathcal{X}_2 \cup \mathcal{Y})$  has no cycles.

To show this, let  $\mathcal{X}_1 \in \mathcal{A}$  and  $\mathcal{Y}$  be partitions of  $S$  such that  $\mathbf{Inc}(S, \mathcal{X}_1 \cup \mathcal{Y})$  has no cycles. We know that there exists a 1-coarsening  $\mathcal{X}_2$  of  $\mathcal{X}_1$  such that  $\mathbf{Inc}(S, \mathcal{X}_2 \cup \mathcal{Y})$  is connected and has no cycles. This 1-coarsening  $\mathcal{X}_2$  is obtained in Step 1. In Step 3, we obtain  $\mathcal{R}_{|\mathcal{X}_2|}$ , and there exists  $\mathcal{X}_3 \in \mathcal{R}_{|\mathcal{X}_2|}$  such that  $\mathbf{Inc}(S, \mathcal{X}_3 \cup \mathcal{Y})$  is connected and has no cycles. Let  $\mathcal{X}_4$  be the partition obtained from  $\mathcal{X}_3$  by taking the original partition before taking a 1-coarsening. We have that  $\mathcal{X}_4 \in \mathcal{R}$  and  $\mathbf{Inc}(S, \mathcal{X}_4 \cup \mathcal{Y})$  has no cycles, as required. By Theorem 4.2,  $|\mathcal{R}| \leq \sum_{1 \leq i \leq |S|} |\mathcal{R}_i| \leq |S| \cdot 2^{|S|-1}$  and Algorithm 1 runs in time  $\mathcal{A}^{\mathcal{O}(1)} 2^{\mathcal{O}(|S|)}$ .  $\square$

## 5 Bounded $\mathcal{P}$ -Block Vertex Deletion

In this section, we prove Theorem 1.1, restated below.

**Theorem 1.1.** *Let  $\mathcal{P}$  be a class of graphs that is block-hereditary, recognizable in polynomial time, and consists of only chordal graphs. Then BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION can be solved in time  $2^{\mathcal{O}(wd^2)}k^2n$  on graphs with  $n$  vertices and treewidth  $w$ .*

We provide an overview of our approach for Theorem 1.1.

1. We first focus on  $S$ -blocks of a  $d$ -labeled  $\mathcal{P}$ -block boundaried graph  $(G, S)$ , which will be the graph that remains after removing some partial solution in the dynamic programming algorithm. For each non-trivial block of  $G[S]$ , we guess its final shape as a  $d$ -labeled biconnected graph, and store the labelings of the vertices and their neighbors in the  $S$ -block of  $G$  containing it. Collectively, we call this information a *characteristic* of  $(G, S)$ .
2. Suppose  $(H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block boundaried graph compatible with  $(G, S)$  such that every  $S$ -block of  $(G, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph. Note that  $(G, S) \oplus (H, S)$  still may have a chordless cycle, and by Proposition 3.7,  $(G, S) \oplus (H, S)$  is chordal if and only if  $\mathbf{Aux}(G, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. If  $(G, S) \oplus (H, S)$  is chordal, then it is easy to check that every block of  $(G, S) \oplus (H, S)$  is contained in one of  $G$  and  $H$ , or an  $S$ -block. Thus, instead of storing  $\mathbf{Aux}(G, S)$ , instead we will store the corresponding partition of the set of components of  $G[S]$ . To avoid storing all such partitions, whose total size might be  $2^{c \cdot w \log w}$  for some constant  $c$ , we use the representative set technique discussed in Section 4.
3. We formally describe and prove an equivalence between two boundaried graphs in Theorem 5.1.

For convenience, we fix an integer  $d \geq 2$  and a class  $\mathcal{P}$  of graphs that is block-hereditary, recognizable in polynomial time, and consists of only chordal graphs. Let  $\mathcal{U}_d$  be the set of all  $d$ -labeled biconnected  $\mathcal{P}$ -block graphs, where each  $H$  in  $\mathcal{U}_d$  has labeling  $L_H$ . For a boundaried graph  $(G, S)$ , we denote by  $\text{Block}(G, S)$  the set of all non-trivial blocks in  $G[S]$ .

### 5.1 Characteristics

For a  $d$ -labeled graph  $(G, S)$  with a labeling  $L$ , a *characteristic* of  $(G, S)$  is a pair  $(g, h)$  of functions  $g : \text{Block}(G, S) \rightarrow \mathcal{U}_d$  and  $h : \text{Block}(G, S) \rightarrow 2^{[d]}$  satisfying the following, for each  $B \in \text{Block}(G, S)$  and the unique  $S$ -block  $H$  of  $G$  containing  $B$ ,

- (a) (label-isomorphic condition)  $H$  is partially label-isomorphic to  $g(B)$ ;
- (b) (coincidence condition) for every  $B' \in \text{Block}(G, S)$  where  $B'$  is contained in  $H$ ,  $g(B') = g(B)$ ;
- (c) (neighborhood condition)  $h(B) = L(N_H(V(B)) \setminus S)$ ; and
- (d) (complete condition) for every  $w$  where  $w \in V(H) \setminus S$  or  $\{w\} = V(H) \cap V(C)$  for some component  $C$  of  $G[S]$ ,  $H[N_H[w]]$  is label-isomorphic to  $g(B)[N_{g(B)}[w]]$  where  $w$  is the vertex in  $g(B)$  with label  $L(w)$ .

For a  $d$ -labeled  $\mathcal{P}$ -block graph with characteristic  $(g, h)$ , the sum  $(G, S) \oplus (H, S)$  respects  $(g, h)$  if for each  $B \in \text{Block}(G, S)$ , the  $S$ -block of  $(G, S) \oplus (H, S)$  containing  $B$  is label-isomorphic to  $g(B)$ .

The following is the main combinatorial result regarding characteristics.

**Theorem 5.1.** *Let  $(G_1, S)$ ,  $(G_2, S)$ , and  $(H, S)$  be  $d$ -labeled  $\mathcal{P}$ -block graphs such that*

- *for each  $i \in \{1, 2\}$ ,  $(G_i, S)$  is compatible with  $(H, S)$ ,*
- *$(G_1, S)$  and  $(G_2, S)$  have the same characteristic  $(g, h)$ , and*
- *$\mathbf{Aux}(G_2, S) \oplus \mathbf{Aux}(H, S)$  has no cycles.*

*If  $(G_1, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph that respects  $(g, h)$ , then  $(G_2, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph that respects  $(g, h)$ .*

*Proof.* Suppose  $(G_1, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph that respects  $(g, h)$ . We first show  $(G_2, S) \oplus (H, S)$  respects  $(g, h)$ . Choose a non-trivial block  $B$  of  $G_2[S]$ , let  $Q := g(B)$ , and let  $F$  be the  $S$ -block of  $(G_2, S) \oplus (H, S)$  containing  $B$ . As a shortcut, set  $S_F := V(F) \cap S$ . Let  $L_F$  be the function from  $V(F)$  to  $[d]$  that sends each vertex to its label from  $G_2$  or  $H$ . Let  $L_Q$  be the labeling of  $Q$ .

We claim that  $L_F$  is a  $d$ -labeling of  $F$ , and  $F$  is label-isomorphic to  $Q$ . We verify the conditions of Proposition 3.1 by regarding  $F$  as the sum of  $(F \cap G_2, S_F)$  and  $(F \cap H, S_F)$  to show that  $F$  is partially label-isomorphic to  $Q$ . We additionally show that  $L_Q(V(Q)) \subseteq L_F(V(F))$ , in order to complete the proof.

**Claim 5.1.1.** *For every non-trivial block  $B'$  of  $G_2[S]$  with  $V(B') \subseteq V(F)$ ,  $g(B') = Q$ .*

*Proof.* Note that  $\mathbf{Aux}(G_2, S) \oplus \mathbf{Aux}(H, S)$  has no cycles. Since  $(g, h)$  is a characteristic of  $(G_2, S)$ , for non-trivial blocks  $B_1, B_2$  of  $G_2[S]$  contained in the same  $S$ -block of  $G_2$ ,  $g(B_1) = g(B_2)$ . Also, since  $(G_1, S) \oplus (H, S)$  respects  $(g, h)$ , for non-trivial blocks  $B_1, B_2$  of  $G_2[S]$  contained in the same  $S$ -block of  $H$ ,  $g(B_1) = g(B_2)$ . Thus, the claim follows from Lemma 3.5.  $\diamond$

Since  $\mathbf{Aux}(G_2, S) \oplus \mathbf{Aux}(H, S)$  has no cycles, by Lemma 3.6,  $\mathbf{Aux}(F \cap G_2, S_F) \oplus \mathbf{Aux}(F \cap H, S_F)$  has no cycles. To apply Proposition 3.1, it remains to show that  $(F \cap G_2, S_F)$  and  $(F \cap H, S_F)$  are block-wise  $Q$ -compatible.

**Claim 5.1.2.**  *$F \cap G_2$  and  $F \cap H$  are block-wise  $Q$ -compatible.*

*Proof.* By Claim 5.1.1 and the fact that  $(g, h)$  is a characteristic of  $(G_2, S)$ ,  $F \cap G_2$  is block-wise partially label-isomorphic to  $Q$ . By Claim 5.1.1 and the fact that  $(G_1, S) \oplus (H, S)$  respects  $(g, h)$ ,  $F \cap H$  is block-wise partially label-isomorphic to  $Q$ .

To confirm the second condition of being block-wise  $Q$ -compatible, let  $B \in \text{Block}(F, S_F)$  and let  $B_1$  and  $B_2$  be the  $S$ -blocks of  $G_2$  and  $H$  containing  $B$  respectively. Let  $B'_1$  be the  $S$ -block of  $G_1$  containing  $B$ . Since  $(G_1, S) \oplus (H, S)$  respects  $(g, h)$ ,  $N_{B'_1}(V(B)) \setminus S$  and  $N_{B_2}(V(B)) \setminus S$  have disjoint sets of labels. As  $(G_1, S)$  and  $(G_2, S)$  have the same characteristic,  $N_{B'_1}(V(B)) \setminus S$  and  $N_{B_1}(V(B)) \setminus S$  have the same set of labels, and thus  $N_{B_1}(V(B)) \setminus S$  and  $N_{B_2}(V(B)) \setminus S$  have disjoint sets of labels. Furthermore, for  $\ell_1 \in L_F(N_{B_1}(V(B)) \setminus S)$  and  $\ell_2 \in L_F(N_{B_2}(V(B)) \setminus S)$ , the vertices in  $Q$  with labels  $\ell_1$  and  $\ell_2$  are not adjacent because there are no edges between  $N_{B'_1}(V(B)) \setminus S$  and  $N_{B_2}(V(B)) \setminus S$  in  $(G_1, S) \oplus (H, S)$ .  $\diamond$



By Claim 5.1.2 and Proposition 3.1,  $L_F$  is a  $d$ -labeling of  $F$  and  $F$  is partially label-isomorphic to  $Q$ . Lastly, we show that  $F$  and  $Q$  have the same set of labels.

**Claim 5.1.3.**  $L_Q(V(Q)) \subseteq L_F(V(F))$ .

*Proof.* Suppose there is a vertex  $v$  in  $Q$  such that  $F$  has no vertex with label  $L_Q(v)$ . We choose such a vertex  $v$  so that there exists  $w \in V(Q)$  that is adjacent to  $v$  in  $Q$  where the label of  $w$  appears in  $F$ . We can choose such vertices  $v$  and  $w$  because  $Q$  is connected,  $V(F) \neq \emptyset$ , and  $L_F(V(F)) \subseteq L_Q(V(Q))$ . Let  $w'$  be the vertex in  $F$  with label  $L_Q(w)$ .

First assume  $w' \in V(F) \setminus S$ . If  $w' \in V(G_2) \setminus S$ , then by the complete condition of the characteristic,  $U[N_U[w']]$  is label-isomorphic to  $Q[N_Q[w]]$ , where  $U$  is the  $S$ -block of  $G_2$  containing  $w'$  and  $V(U) \subseteq V(F)$ . If  $w' \in V(H) \setminus S$ , then since  $(G_1, S) \oplus (H, S)$  respects  $(g, h)$ ,  $U[N_U[w']]$  is label-isomorphic to  $Q[N_Q[w]]$ , where  $U$  is the  $S$ -block of  $H$  containing  $w'$  and  $V(U) \subseteq V(F)$ . Thus, in this case,  $F$  contains a vertex with label  $L_Q(v)$ ; a contradiction. We may assume that  $w'$  is contained in  $S$ .

Next, we assume that  $\{w'\}$  is the vertex set of some component of  $F[S_F]$ . As  $F$  has at least 3 vertices,  $w'$  has a neighbor in  $F$ . We claim that  $w'$  has neighbors in precisely one of  $F \cap G_2$  and  $F \cap H$ . Towards a contradiction, suppose  $w'$  has neighbors in both  $F \cap G_2$  and  $F \cap H$ . Note that  $F - w'$  is connected. We take a shortest path  $P$  from  $N_{F \cap G_2}(w')$  to  $N_{F \cap H}(w')$ . By construction, the end vertices of  $P$  are not adjacent, and  $w'$  is not adjacent to any internal vertices of  $P$ . Thus,  $F[\{w'\} \cup V(P)]$  is a chordless cycle, contradicting the fact that  $F$  is partially label-isomorphic to  $Q$  and  $Q$  is chordal. We conclude that  $w'$  has neighbors in precisely one of  $F \cap G_2$  and  $F \cap H$ .

If  $w'$  has a neighbor in  $F \cap G_2$ , then by the complete condition of the characteristic,  $U[N_U[w']]$  is label-isomorphic to  $Q[N_Q[w]]$ , where  $U$  is the  $S$ -block of  $G_2$  containing  $w'$  and  $V(U) \subseteq V(F)$ . If  $w'$  has a neighbor in  $F \cap H$ , then since  $(G_1, S) \oplus (H, S)$  respects  $(g, h)$ ,  $U[N_U[w']]$  is label-isomorphic to  $Q[N_Q[w]]$ , where  $U$  is the  $S$ -block of  $H$  containing  $w'$  and  $V(U) \subseteq V(F)$ . Thus, in this case,  $F$  contains a vertex with label  $L_Q(v)$ ; a contradiction.

Finally, we may assume that there is a non-trivial block  $B'$  of  $F[S_F]$  containing  $w'$ . We observe that the  $S$ -block of  $(G_1, S) \oplus (H, S)$  containing  $B'$  is label-isomorphic to  $Q$ . We also observe that every label appearing in the neighborhood of  $w'$  in the  $S$ -block of  $(G_1, S) \oplus (H, S)$  containing  $B'$  appears in the neighborhood of  $w'$  in  $(G_2, S) \oplus (H, S)$  as well, because  $(G_1, S)$  and  $(G_2, S)$  have the same characteristic. This contradicts the assumption that  $F$  has no vertex with label  $L_Q(v)$ . We conclude that  $L_Q(V(Q)) \subseteq L_F(V(F))$ .  $\diamond$

We conclude that  $F$  is label-isomorphic to  $Q$ . Since  $B$  was arbitrarily chosen, this implies that  $(G_2, S) \oplus (H, S)$  respects  $(g, h)$ . Lastly, we confirm that  $(G_2, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph.

**Claim 5.1.4.** *The graph  $(G_2, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph.*

*Proof.* It is sufficient to show that every non  $S$ -block of  $(G_2, S) \oplus (H, S)$  is fully contained in  $G_2$  or  $H$ . We first observe that since  $\mathbf{Aux}(G_2, S) \oplus \mathbf{Aux}(H, S)$  has no cycles and every  $S$ -block of  $(G_2, S) \oplus (H, S)$  is chordal, by Proposition 3.7,  $(G_2, S) \oplus (H, S)$  is chordal.

Suppose towards a contradiction that there is a non  $S$ -block  $U$  of  $(G_2, S) \oplus (H, S)$  intersecting both  $G_2 - S$  and  $H - S$ . We choose a triple  $(v, w, D)$  such that

- $v \in V(U) \cap (V(G_2) \setminus S)$ ,  $w \in V(U) \cap (V(H) \setminus S)$ ,  $D$  is a cycle containing  $v$  and  $w$  in  $U$ ; and
- the length of  $D$  is minimum.

Let  $P_1$  and  $P_2$  be the two paths from  $v$  to  $w$  in  $D$ .

We claim that there are no edges between the internal vertices of  $P_1$  and the internal vertices of  $P_2$ . Suppose there is an edge  $p_1 p_2$  for some  $p_1 \in V(P_1) \setminus \{v, w\}$  and  $p_2 \in V(P_2) \setminus \{v, w\}$ . One of  $p_1$  and  $p_2$  is contained in  $G_2 - S$  or  $H - S$ , as  $U$  can contain at most one vertex of each connected component of  $G_2[S]$ . Now, if  $p_1$  and  $p_2$  are contained in  $G_2$ , then we can replace  $v$  with one of  $p_1$  and  $p_2$  that is in  $G_2 - S$ , and obtain a cycle shorter than  $D$ ; a contradiction. Similarly, if they are contained in  $H$ , then we obtain a cycle shorter than  $D$ . This implies that there are no edges between the internal vertices of  $P_1$  and the internal vertices of  $P_2$ . Since  $v$  is not adjacent to  $w$ ,  $D$  is a chordless cycle, which contradicts the fact that  $(G_2, S) \oplus (H, S)$  is chordal. We conclude that every non  $S$ -block of  $(G_2, S) \oplus (H, S)$  is fully contained in  $G_2$  or  $H$ , and therefore  $(G_2, S) \oplus (H, S)$  is a  $d$ -labeled  $\mathcal{P}$ -block graph.  $\diamond$

This concludes the proof.  $\square$

## 5.2 Main algorithm

Let  $(G, S)$  be a boundaried graph, and let  $\mathcal{C}$  be the set of components of  $G[S]$ . When  $\mathcal{P}$  is the partition of  $\mathcal{C}$  such that two components of  $G[S]$  are in the same part if and only if they are in the same component of  $G$ , we denote this as  $\mathbf{Inc}(\mathcal{C}, \mathcal{P}) \sim \mathbf{Aux}(G, S)$ . One can observe that there is an isomorphism from  $\mathbf{Inc}(\mathcal{C}, \mathcal{P})$  to  $\mathbf{Aux}(G, S)$  that maps each component of  $\mathcal{C}$  to the same component.

*Proof of Theorem 1.1.* Using Theorem 2.1 and Lemma 2.2, we obtain a nice tree decomposition of  $G$  of width at most  $5w + 4$  in time  $\mathcal{O}(c^w \cdot n)$  for some constant  $c$ . Let  $(T, \mathcal{B} = \{B_t\}_{t \in V(T)})$  be the resulting nice tree decomposition and let  $r$  be its root node. For each node  $t$  of  $T$ , let  $G_t$  be the subgraph of  $G$  induced by the union of all bags  $B_{t'}$  where  $t'$  is a descendant of  $t$ . Recall that  $\mathcal{U}_d$  is the class of all biconnected  $d$ -labeled  $\mathcal{P}$ -block graphs, where each  $H$  in  $\mathcal{U}_d$  has labeling  $L_H$ . Note that  $|\mathcal{U}_d| \leq 2^{\binom{d}{2}}$ . We define the following notation for every pair consisting of a node  $t$  of  $T$  and  $X \subseteq B_t$ :

1. Let  $\text{Comp}(t, X)$  be the set of all connected components of  $G[B_t \setminus X]$ .
2. Let  $\text{Part}(t, X)$  be the set of all partitions of  $\text{Comp}(t, X)$ .
3. Let  $\text{Block}(t, X)$  be the set of all non-trivial blocks of  $G[B_t \setminus X]$ .

For each node  $t$  of  $T$ ,  $X \subseteq B_t$ , and a function  $L : B_t \setminus X \rightarrow [d]$ , we define  $\mathcal{F}(t, X, L)$  as the set of all pairs  $(g, h)$  consisting of functions  $g : \text{Block}(t, X) \rightarrow \mathcal{U}_d$  and  $h : \text{Block}(t, X) \rightarrow 2^{[d]}$ . We say that  $(g, h)$  is *valid* if

- $L$  is a  $d$ -labeling of  $G[B_t \setminus X]$ ,
- for each  $B \in \text{Block}(t, X)$ ,  $B$  is partially label-isomorphic to  $g(B)$ , and
- for each  $B \in \text{Block}(t, X)$ ,  $L(V(B)) \cap h(B) = \emptyset$ .

Furthermore, for  $i \in \{0, 1, \dots, k\}$  and  $(g, h) \in \mathcal{F}(t, X, L)$ , let  $c[t, (X, L, i, (g, h))]$  be the family of all partitions  $\mathcal{X}$  in  $\text{Part}(t, X)$  satisfying the following property: there exist  $S \subseteq V(G_t) \setminus B_t$  with  $|S| = i$  and a  $d$ -labeling  $L'$  of  $G_t - (X \cup S)$  where

- $L = L'|_{B_t \setminus X}$ ,
- $G_t - (X \cup S)$  is a  $\mathcal{P}$ -block graph,
- $(g, h)$  is a characteristic of  $(G_t - (X \cup S), B_t \setminus X)$ , and
- $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}) \sim \mathbf{Aux}(G_t - (X \cup S), B_t \setminus X)$ .

Such a pair  $(S, L')$  is called a *partial solution* with respect to  $c[t, (X, L, i, (g, h))]$  and  $\mathcal{X}$ . If the tuple is clear from the context, we simply say that  $(S, L')$  is a partial solution with respect to  $\mathcal{X}$ . It is easy to verify that  $c[t, (X, L, i, (g, h))] = \emptyset$  if  $(g, h)$  is not valid. Let  $\mathcal{M}_t$  be the set of all possible tuples  $(X, L, i, (g, h))$  at node  $t$ .

The main idea of the algorithm is that instead of fully computing  $c[t, M]$  for  $M = (X, L, i, (g, h)) \in \mathcal{M}_t$ , we recursively enumerate a set  $r[t, M]$  that represents partial solutions for  $c[t, M]$ . Formally, for a subset  $r[t, M] \subseteq c[t, M]$ , we denote  $r[t, M] \equiv c[t, M]$  if

- for every  $\mathcal{X} \in c[t, M]$  and a partial solution  $(S, L')$  with respect to  $\mathcal{X}$  and  $S_{out} \subseteq V(G) \setminus V(G_t)$  where  $G - (S \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ , there exists  $\mathcal{X}_1 \in r[t, M]$  and a partial solution  $(S', L'')$  with respect to  $\mathcal{X}_1$  such that  $G - (S' \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ .

By the definition of  $r[t, M]$ , the problem is a YES-instance if and only if there exists  $(X, L, i, (g, h)) \in \mathcal{M}_r$  with  $|X| + i \leq k$  such that  $r[r, (X, L, i, (g, h))] \neq \emptyset$ . To decide whether the problem is a YES-instance, we enumerate  $r[t, M]$  for all nodes  $t$  and all  $M \in \mathcal{M}_t$ .

Whenever we update  $r[t, M]$ , we confirm that  $|r[t, M]| \leq w \cdot 2^{w-1}$ . This is a consequence of Proposition 4.3. We describe how to update families  $r[t, M]$  depending on the type of node  $t$ , and prove the correctness of each procedure. We fix such a tuple. For each leaf node  $t$ , we assign  $r[t, (\emptyset, L, i, (g, h))] := \emptyset$  where  $L, g, h$  are empty functions. We may assume  $t$  is not a leaf node. Let  $M := (X, L, i, (g, h)) \in \mathcal{M}_t$ . We may assume  $(g, h)$  is valid.

### 1) $t$ is an introduce node with child $t'$ :

Let  $v$  be the vertex in  $B_t \setminus B_{t'}$ . If  $v \in X$ , then  $G_t - X = G_{t'} - (X \setminus \{v\})$  and  $B_t \setminus X = B_{t'} \setminus (X \setminus \{v\})$ . So, we can set  $r[t, M] := r[t', (X \setminus \{v\}, L, i, (g, h))]$ . We assume  $v \notin X$ , and let  $L_{res} := L|_{B_{t'} \setminus X}$ .

For a pair  $(g, h) \in \mathcal{F}(t, X, L)$ , a pair  $(g', h') \in \mathcal{F}(t', X, L_{res})$  is called the *restriction* of  $(g, h)$  if

- for  $B_1 \in \text{Block}(t', X)$  and  $B_2 \in \text{Block}(t, X)$  with  $V(B_1) \subseteq V(B_2)$ ,  $g'(B_1) = g(B_2)$ , and if  $v \in V(B_2)$ , then every vertex in  $g'(B_1)$  with label in  $h'(B_1)$  is not adjacent to the vertex in  $g'(B_1)$  with label  $L(v)$ ,
- for  $B_1 \in \text{Block}(t', X)$  and  $B_2 \in \text{Block}(t, X)$  with  $V(B_1) \subseteq V(B_2)$  and  $v \notin V(B_2)$ ,  $h'(B_1) = h(B_2)$ , and
- for  $B_2 \in \text{Block}(t, X)$  containing  $v$ ,  $h(B_2) = \bigcup_{B_1 \in \text{Block}(t', X), V(B_1) \subseteq V(B_2)} h(B_1)$ .

**Claim 5.1.5.** *For every  $\mathcal{X} \in \text{Part}(t, X)$ ,  $\mathcal{X} \in c[t, M]$  if and only if there exist a restriction  $(g', h')$  of  $(g, h)$  and  $\mathcal{Y} \in c[t', (X, L_{res}, i, (g', h'))]$  such that*

- $v$  has neighbors on at most one component in each part of  $\mathcal{Y}$  (that is,  $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \oplus \mathbf{Aux}(G[B_t \setminus X], B_{t'} \setminus X)$  has no cycles), and
- if  $v$  has at least one neighbor in  $G[B_t \setminus X]$ , then  $\mathcal{X}$  is the partition obtained from  $\mathcal{Y}$  by, for parts  $Y_1, \dots, Y_m$  of  $\mathcal{Y}$  containing components having a neighbor of  $v$ , removing all of  $Y_1, \dots, Y_m$  and adding a part that consists of all components of  $G[B_t \setminus X]$  that are not contained in parts of  $\mathcal{Y} \setminus \{Y_1, \dots, Y_m\}$ ; and otherwise,  $\mathcal{X} = \mathcal{Y} \cup \{\{v\}\}$ .

*Proof.* Suppose  $\mathcal{X} \in c[t, M]$  and let  $(S, L_t)$  be a partial solution with respect to  $\mathcal{X}$ . Let  $\mathcal{Y} \in \text{Part}(t', X)$  such that  $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \sim \mathbf{Aux}(G_{t'} - (X \cup S), B_{t'} \setminus X)$ . As

$$G_t - (X \cup S) = (G_{t'} - (X \cup S), B_{t'} \setminus X) \oplus (G[B_t \setminus X], B_{t'} \setminus X)$$

and  $G_t - (X \cup S)$  is chordal, by Proposition 3.7,

$$\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \oplus \mathbf{Aux}(G[B_t \setminus X], B_{t'} \setminus X)$$

has no cycles. The second condition holds by the definition of  $\mathcal{Y}$ . Since we can naturally obtain a restriction  $(g', h')$  of  $(g, h)$  for  $(G_{t'} - (X \cup S), B_{t'} \setminus X)$ , this concludes the proof of the forward direction.

For the converse, suppose there exist  $(g', h')$  and  $\mathcal{Y}$  satisfying the assumption. Let  $M_{res} := (X, L_{res}, i, (g', h'))$ , and let  $(S, L_{t'})$  be a partial solution with respect to  $c[t', M_{res}]$  and  $\mathcal{Y}$ . For convenience, let  $H := G_t - (X \cup S)$  and  $H' := G_{t'} - (X \cup S)$ . Let  $L_t : V(H) \rightarrow [d]$  be the function obtained from  $L_{t'}$  by further assigning  $L_t(v) := L(v)$ .

We claim that  $(g, h)$  is a characteristic of  $(H, B_t \setminus X)$ . Before checking the conditions of a characteristic, we show that if two non-trivial blocks  $D_1$  and  $D_2$  of  $G[B_t \setminus X]$  are contained in the same  $(B_{t'} \setminus X)$ -block of  $H$ , then  $g'(D_1) = g'(D_2)$ . For  $D_1, D_2 \in \text{Block}(t', X)$ , if  $D_1$  and  $D_2$  are contained in the same  $(B_{t'} \setminus X)$ -block of  $(H', B_{t'} \setminus X)$ , then  $g'(D_1) = g'(D_2)$  because  $(g', h')$  is a characteristic of  $(H', B_{t'} \setminus X)$ . Also, if  $D_1$  and  $D_2$  are contained in the same  $(B_{t'} \setminus X)$ -block of  $(G_t[B_t \setminus X], B_{t'} \setminus X)$ , then  $g'(D_1) = g'(D_2)$  as  $(g', h')$  is a restriction of  $(g, h)$ . By the assumption,  $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \oplus \mathbf{Aux}(G[B_t \setminus X], B_{t'} \setminus X)$  and equivalently,  $\mathbf{Aux}(H', B_{t'} \setminus X) \oplus \mathbf{Aux}(G[B_t \setminus X], B_{t'} \setminus X)$  have no cycles. Therefore, by Lemma 3.5, if  $D_1$  and  $D_2$  are contained in the same  $(B_{t'} \setminus X)$ -block of  $H$ , then  $g'(D_1) = g'(D_2)$ .

Let  $B \in \text{Block}(t, X)$  and  $F$  be the  $(B_t \setminus X)$ -block of  $H$  containing  $B$ .

1. (Coincidence condition)

Let  $B' \in \text{Block}(H, B_t \setminus X)$  such that  $B \neq B'$  and  $B'$  is contained in  $F$ . If  $|V(B) \setminus \{v\}| = 1$  or  $|V(B') \setminus \{v\}| = 1$ , then  $B$  and  $B'$  cannot be contained in the same  $(B_t \setminus X)$ -block of  $H$ . Thus, both  $B$  and  $B'$  contain non-trivial blocks  $U$  and  $U'$  in  $G[B_{t'} \setminus X]$  respectively, where  $g'(U) = g'(U')$ . This implies that  $g(B) = g(B')$ .

2. (Neighborhood condition)

It follows from the assumption that  $(g', h')$  is a restriction of  $(g, h)$ . If  $B$  is a block of  $G[B_{t'} \setminus X]$ , then  $h(B) = h'(B)$ . Otherwise  $B$  contains  $v$ , and every neighbor of  $B$  in  $F$  has a neighbor in some block  $B_1 \in \text{Block}(t', X)$  where  $V(B_1) \subseteq V(B)$ . Thus, we have  $h(B) = \bigcup_{B_1 \in \text{Block}(t', X), V(B_1) \subseteq V(B)} h(B_1)$ .

### 3. (Label-isomorphic condition)

We prove that  $F$  is partially label-isomorphic to  $g(B)$ . Let  $F_1 := F \cap H'$ ,  $F_2 := F \cap G[B_t \setminus X]$ , and  $U = V(F_1) \cap V(F_2)$ . Since  $\mathbf{Aux}(H', B_{t'} \setminus X) \oplus \mathbf{Aux}(G[B_t \setminus X], B_{t'} \setminus X)$  has no cycles, by Lemma 3.6,  $\mathbf{Aux}(F_1, U) \oplus \mathbf{Aux}(F_2, U)$  has no cycles. To apply Proposition 3.1, we verify that  $(F_1, U)$  and  $(F_2, U)$  are block-wise  $g(B)$ -compatible. We observed that if two non-trivial blocks  $D_1$  and  $D_2$  of  $G[B_{t'} \setminus X]$  are contained in  $F$ , then  $g'(D_1) = g'(D_2) = g(B)$ . Since  $(g', h')$  is a characteristic of  $(H', B_{t'} \setminus X)$ ,  $(F_1, U)$  is block-wise partially label-isomorphic to  $g(B)$ . Also, since  $(g, h)$  is valid,  $(F_2, U)$  is block-wise partially label-isomorphic to  $g(B)$ . As  $(g', h')$  is a restriction of  $(g, h)$ , for  $B_1 \in \text{Block}(t', X)$  and  $B_2 \in \text{Block}(t, X)$  with  $V(B_1) \subseteq V(B_2)$  and  $v \in V(B_2)$ , every vertex in  $g'(B_1)$  with label in  $h'(B_1)$  is not adjacent to the vertex in  $g'(B_1)$  with label  $L(v)$ . Because of this condition, the second condition of being block-wise  $g(B)$ -compatible is also satisfied.

By Proposition 3.1,  $F$  is partially label-isomorphic to  $g(B)$ .

### 4. (Complete condition)

This follows from the fact that  $(g', h')$  is a restriction of  $(g, h)$  and it is a characteristic of  $(H', B_{t'} \setminus X)$ .

All together we can conclude that  $\mathcal{X} \in c[t, M]$ . ◇

When  $v \notin X$ , we update  $r[t, M]$  as follows. Set  $\mathcal{K} := \emptyset$ . For every  $(g', h') \in \mathcal{F}(t', X, L_{res})$ , we test whether  $(g', h')$  is a restriction of  $(g, h)$ . Assume  $(g', h')$  is a restriction of  $(g, h)$ , otherwise, we skip it. Now, for each  $\mathcal{Y} \in r[t', (X, L_{res}, i, (g', h'))]$ , we check the two conditions for  $(g', h')$  and  $\mathcal{Y}$  in Claim 5.1.5, and if they are satisfied, then we add the set  $\mathcal{X}$  described in Claim 5.1.5 to  $\mathcal{K}$ ; otherwise, we skip it. Since  $|\mathcal{F}(t', X, L_{res})| \leq 2^{\mathcal{O}(wd^2)}$  and  $|r[t', (X, L_{res}, i, (g', h'))]| \leq w \cdot 2^{w-1}$ , the whole procedure can be done in time  $2^{\mathcal{O}(wd^2)}$ . After we do this for all possible candidates, we take a representative set of  $\mathcal{K}$  using Proposition 4.3, and assign the resulting set to  $r[t, M]$ . Since  $|\mathcal{K}| \leq 2^{\mathcal{O}(wd^2)}$ , we can apply Proposition 4.3 in time  $2^{\mathcal{O}(wd^2)}$ . Also, we have  $|r[t, M]| \leq w \cdot 2^{w-1}$ .

We claim that  $r[t, M] \equiv c[t, M]$ . Let  $G_{out} := G - (V(G_t) \setminus B_t)$ . Let  $\mathcal{X} \in c[t, M]$  and  $(S, L')$  be a partial solution with respect to  $\mathcal{X}$ , and suppose there exists  $S_{out} \subseteq V(G) \setminus V(G_t)$  where

$$G - (S \cup X \cup S_{out}) = (G_t - (X \cup S), B_t \setminus X) \oplus (G_{out} - (X \cup S_{out}), B_t \setminus X)$$

is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . Note that every  $(B_{t'} \setminus X)$ -block of  $G - (S \cup X \cup S_{out})$  is chordal as such a block is also a  $(B_t \setminus X)$ -block of  $G - (S \cup X \cup S_{out})$ . Since  $G - (S \cup X \cup S_{out})$  is chordal, by Proposition 3.7,  $\mathbf{Aux}(G_{t'} - (X \cup S), B_{t'} \setminus X) \oplus \mathbf{Aux}(G_{out} - (X \cup S_{out}), B_{t'} \setminus X)$  has no cycles. Recall that  $M_{res} := (X, L_{res}, i, (g', h'))$ . As  $r[t', M_{res}] \equiv c[t', M_{res}]$ , there exist  $\mathcal{Y} \in r[t', M_{res}]$  and a partial solution  $(S', L'')$  with respect to  $\mathcal{Y}$  such that  $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \sim \mathbf{Aux}(G_{t'} - (X \cup S'), B_{t'} \setminus X)$ , and  $\mathbf{Aux}(G_{t'} - (X \cup S'), B_{t'} \setminus X) \oplus \mathbf{Aux}(G_{out} - (X \cup S_{out}), B_{t'} \setminus X)$  has no cycles. By Theorem 5.1,  $G - (S' \cup X \cup S_{out})$  is also a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ .

By the update procedure, the partition  $\mathcal{X}_1$  where  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1) \sim \mathbf{Aux}(G_t - (X \cup S'), B_t \setminus X)$  is added to the set  $\mathcal{K}$ , and there exist  $\mathcal{X}_2 \in r[t, M]$  and a partial solution  $(S'', L''')$  with respect to  $\mathcal{X}_2$  such that  $G - (S'' \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . This shows that  $r[t, M] \equiv c[t, M]$ .

## 2) $t$ is a forget node with child $t'$ :

Let  $v$  be the vertex in  $B_{t'} \setminus B_t$ . For an extension  $L'$  of  $L$  on  $B_{t'} \setminus X$ , a pair  $(g', h') \in \mathcal{F}(t', X, L')$  is called an *extension* of  $(g, h)$  with respect to  $L'$  if

- for  $B_1 \in \text{Block}(t, X)$  and  $B_2 \in \text{Block}(t', X)$  with  $V(B_1) \subseteq V(B_2)$ ,
  - $g'(B_2) = g(B_1)$ ,
  - if  $v \notin V(B_2)$ , then  $h'(B_2) = h(B_1)$ ,
  - if  $v \in V(B_2)$ , then  $L'(v) \in h(B_1)$ ; and
- for  $B_2 \in \text{Block}(t', X)$  containing  $v$ ,  $h'(B_2) = \bigcup_{B_1 \in \text{Block}(t, X), V(B_1) \subseteq V(B_2)} h(B_1) \setminus \{L'(v)\}$ .

We show the following:

**Claim 5.1.6.** *For every  $\mathcal{X} \in \text{Part}(t, X)$ ,  $\mathcal{X} \in c[t, M]$  if and only if one of the following holds:*

1.  $\mathcal{X} \in c[t', (X \cup \{v\}, L, i - 1, (g, h))]$ , or
2. there exists an extension  $L_{ext}$  of  $L$  on  $B_{t'} \setminus X$ , an extension  $(g', h')$  of  $(g, h)$  in  $\mathcal{F}(t', X, L_{ext})$  with respect to  $L_{ext}$ , and  $\mathcal{Y} \in c[t', (X, L_{ext}, i, (g', h'))]$  such that  $\mathcal{X}$  is the partition obtained from  $\mathcal{Y}$  by replacing the component  $U$  of  $B_{t'} \setminus X$  containing  $v$  with the components of  $B_t \setminus X$  contained in  $U$ .

*Proof.* We first show the backward direction. If  $\mathcal{X} \in c[t', (X \cup \{v\}, L, i - 1, (g, h))]$ , then  $\mathcal{X} \in c[t, M]$ , as we can put  $v$  into the partial solution. Suppose statement 2 holds. Then there exists a partial solution  $(S, L')$  with respect to  $c[t', (X, L_{ext}, i, (g', h'))]$  and  $\mathcal{Y}$ . It is not difficult to verify that  $(g, h)$  is the characteristic of  $(G_t - (X \cup S), B_t \setminus X)$  and  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{Y}) \sim \mathbf{Aux}(G_t - (X \cup S), B_t \setminus X)$ . Thus,  $\mathcal{X} \in c[t, M]$ .

For the other direction, suppose  $\mathcal{X} \in c[t, M]$ , and let  $(S, L')$  be a partial solution with respect to  $\mathcal{X}$ . If  $v \in S$ , then  $\mathcal{X} \in c[t', (X \cup \{v\}, L, i - 1, (g, h))]$ . This corresponds to the first case. We may assume that  $v \notin S$ . Let  $L_{ext} := L'|_{B_{t'} \setminus X}$  and  $\mathcal{Y} \in \text{Part}(t', X)$  such that  $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \sim \mathbf{Aux}(G_t - (X \cup S), B_{t'} \setminus X)$ . Since  $G_t - (X \cup S) = G_{t'} - (X \cup S)$ , one can observe that  $\mathcal{X}$  is the partition obtained from  $\mathcal{Y}$  by replacing the connected component  $U$  of  $B_t \setminus X$  containing  $v$  with the connected components of  $B_{t'} \setminus X$  contained in  $U$ . We focus on showing that there exists an extension  $(g', h')$  of  $(g, h)$  in  $\mathcal{F}(t', X, L_{ext})$  that is the characteristic of  $(G_{t'} - (X \cup S), B_{t'} \setminus X)$ .

We construct  $(g', h')$  as follows.

- Let  $B \in \text{Block}(t', X)$  containing  $v$ . If there exists  $B' \in \text{Block}(t, X)$  where  $B$  and  $B'$  are contained in the same block of  $G_t - (X \cup S)$ , then we let  $g'(B) = g(B')$ . Otherwise, we know that the block of  $G_{t'} - (X \cup S)$  containing  $v$  is label-isomorphic to a graph in  $\mathcal{U}_d$ ; let  $g'(B)$  be this graph.
- For  $B \in \text{Block}(t', X)$  with  $v \notin V(B)$ , let  $g'(B) = g(B)$ .
- Also, for every  $B \in \text{Block}(t', X)$ , let  $h'(B)$  be the set of labels that appear in the neighbors of vertices of  $B$  in the block of  $G_{t'} - (S \cup X)$  containing  $B$ .

Then  $(g', h')$  is an extension of  $(g, h)$ , and  $\mathcal{Y} \in c[t', (X, L_{ext}, i, (g', h'))]$ . ◇

We update  $r[t, M]$  as follows. Set  $\mathcal{K} := \emptyset$ . First, we add all partitions in  $r[t', (X \cup \{v\}, L, i - 1, (g, h))]$  to  $\mathcal{K}$ . At the second step, for every extension  $L_{ext}$  of  $L$  on  $B_{t'} \setminus X$  and every  $(g', h') \in \mathcal{F}(t', X, L_{ext})$ , we test whether  $(g', h')$  is an extension of  $(g, h)$ . In the case when  $(g', h')$  is an extension of  $(g, h)$  with respect to  $L_{ext}$ , for all partitions  $\mathcal{Y} \in r[t', (X, L_{ext}, i, (g', h'))]$ , we add the

set  $\mathcal{X}$  satisfying the second statement in Claim 5.1.6 to  $\mathcal{K}$ , and otherwise, we skip this pair. This can be done in time  $2^{\mathcal{O}(wd^2)}$ . After we do this for all possible candidates, we take a representative set of  $\mathcal{K}$  using Proposition 4.3, and assign the resulting set to  $r[t, M]$ . Notice that  $|\mathcal{K}| \leq 2^{\mathcal{O}(wd^2)}$ . By Proposition 4.3, the procedure of obtaining a representative set can be done in time  $2^{\mathcal{O}(wd^2)}$ , and we have  $|r[t, M]| \leq w \cdot 2^{w-1}$ .

We claim that  $r[t, M] \equiv c[t, M]$ . Let  $\mathcal{X} \in c[t, M]$  and let  $(S, L')$  be a partial solution with respect to  $\mathcal{X}$  and let  $S_{out} \subseteq V(G) \setminus V(G_t)$  where  $G - (S \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . Let  $G_{out} := G - (V(G_{t'}) \setminus B_{t'})$ . The graph  $G - (S \cup X \cup S_{out})$  can be seen as  $(G_{t'} - (X \cup S), B_{t'} \setminus X) \oplus (G_{out} - (X \cup S_{out}), B_{t'} \setminus X)$ .

Note that every  $(B_{t'} \setminus X)$ -block of  $G - (S \cup X \cup S_{out})$  is chordal. Since  $G - (S \cup X \cup S_{out})$  is chordal, by Proposition 3.7,  $\mathbf{Aux}(G_{t'} - (X \cup S), B_{t'} \setminus X) \oplus \mathbf{Aux}(G_{out} - (X \cup S_{out}), B_{t'} \setminus X)$  has no cycles. As  $r[t', (X, L_{ext}, i, (g', h'))] \equiv c[t', (X, L_{ext}, i, (g', h'))]$ , there exists  $\mathcal{Y} \in r[t', (X, L_{ext}, i, (g', h'))]$  and a partial solution  $(S', L'')$  with respect to  $\mathcal{Y}$  such that  $\mathbf{Inc}(\text{Comp}(t', X), \mathcal{Y}) \sim \mathbf{Aux}(G_{t'} - (X \cup S'), B_{t'} \setminus X)$ , and thus  $\mathbf{Aux}(G_{t'} - (X \cup S'), B_{t'} \setminus X) \oplus \mathbf{Aux}(G_{out} - (X \cup S_{out}), B_{t'} \setminus X)$  has no cycles. By Theorem 5.1,  $G - (S' \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g', h')$  for some extension  $(g', h')$  of  $(g, h)$ . By the procedure, the partition  $\mathcal{X}_1$  where  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1) \sim \mathbf{Aux}(G_t - (X \cup S'), B_t \setminus X)$  is added to the set  $\mathcal{K}$ , and there exists  $\mathcal{X}_2 \in r[t, M]$  and a partial solution  $(S'', L''')$  with respect to  $\mathcal{X}_2$  such that  $G - (S'' \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . This shows that  $r[t, M] \equiv c[t, M]$ .

### 3) $t$ is a join node with two children $t_1$ and $t_2$ :

We show the following:

**Claim 5.1.7.** *For every  $\mathcal{X} \in \text{Part}(t, X)$ ,  $\mathcal{X} \in c[t, M]$  if and only if there exist integers  $i_1, i_2$  with  $i_1 + i_2 = i$ ,  $(g, h_1) \in \mathcal{F}(t_1, X, L)$ ,  $(g, h_2) \in \mathcal{F}(t_2, X, L)$ ,  $\mathcal{X}_1 \in c[t_1, (X, L, i_1, (g, h_1))]$ , and  $\mathcal{X}_2 \in c[t_2, (X, L, i_2, (g, h_2))]$  such that*

- $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1 \cup \mathcal{X}_2)$  has no cycles,
- $\mathcal{X} = \mathcal{X}_1 \uplus \mathcal{X}_2$ , and
- for each  $B \in \text{Block}(t, X)$ ,  $h_1(B) \cap h_2(B) = \emptyset$  and  $h(B) = h_1(B) \cup h_2(B)$ , and for  $\ell_1 \in h_1(B)$  and  $\ell_2 \in h_2(B)$ , the vertices with labels  $\ell_1$  and  $\ell_2$  in  $g(B)$  are not adjacent.

*Proof.* The forward direction is straightforward. For the converse direction, suppose there exist integers  $i_1, i_2$  with  $i_1 + i_2 = i$ ,  $(g, h_1)$ ,  $(g, h_2)$ , and partitions  $\mathcal{X}_1, \mathcal{X}_2$  as specified in the statement. For each  $j \in \{1, 2\}$ , let  $M_j := (X, L, i_j, (g, h_j))$  and  $(S_j, L_j)$  be a partial solution with respect to  $c[t_j, M_j]$  and  $\mathcal{X}_j$ . Furthermore, let  $H_j := G_{t_j} - (X \cup S_j)$ , and  $H := H_1 \cup H_2$ , and  $L_H := L_1 \oplus L_2$ . We claim that  $(g, h)$  is a characteristic of  $(H, B_t \setminus X)$ .

#### 1. (Coincidence condition)

Let  $i \in \{1, 2\}$ . Since  $(g, h_i)$  is a characteristic of  $H_i$ , if  $B_1, B_2 \in \text{Block}(t, X)$  are contained in the same  $(B_t \setminus X)$ -block of  $H_i$ ,  $g(B_1) = g(B_2)$ . Since  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1 \cup \mathcal{X}_2)$  and equivalently,  $\mathbf{Aux}(H_1, B_t \setminus X) \oplus \mathbf{Aux}(H_2, B_t \setminus X)$  have no cycles, by Lemma 3.5, if  $B_1, B_2 \in \text{Block}(t, X)$  are contained in the same  $(B_t \setminus X)$ -block of  $H$ ,  $g(B_1) = g(B_2)$ .

#### 2. (Neighborhood condition)

This follows from the assumption that  $h(B) = h_1(B) \cup h_2(B)$  for each  $B \in \text{Block}(t, X)$ .

3. (Label-isomorphic condition)

Let  $B \in \text{Block}(t, X)$  and let  $F$  be the  $(B_t \setminus X)$ -block of  $H$  containing  $B$ . We show that  $F$  is partially label-isomorphic to  $g(B)$ . Let  $U := V(F) \cap (B_t \setminus X)$ .

Since  $\mathbf{Aux}(H_1, B_t \setminus X) \oplus \mathbf{Aux}(H_2, B_t \setminus X)$  has no cycles, by Lemma 3.6,  $\mathbf{Aux}(F \cap H_1, U) \oplus \mathbf{Aux}(F \cap H_2, U)$  has no cycles. Since each  $(g, h_j)$  is a characteristic of  $(H_j, B_{t_j} \setminus X)$ ,  $(F \cap H_1, U)$  and  $(F \cap H_2, U)$  are block-wise partially label-isomorphic to  $g(B)$ . Moreover,  $(F \cap H_1, U)$  and  $(F \cap H_2, U)$  are block-wise  $g(B)$ -compatible, because of the assumption that for each  $B \in \text{Block}(t, X)$ ,  $h_1(B) \cap h_2(B) = \emptyset$  and  $h(B) = h_1(B) \cup h_2(B)$ , and for  $\ell_1 \in h_1(B)$  and  $\ell_2 \in h_2(B)$ , the vertices with labels  $\ell_1$  and  $\ell_2$  in  $g(B)$  are not adjacent. By Proposition 3.1,  $F$  is partially label-isomorphic to  $g(B)$ .

4. (Complete condition)

This follows from the fact that each  $(g, h_j)$  is a characteristic of  $(H_j, B_{t_j} \setminus X)$ .

This proves that  $(g, h)$  is a characteristic of  $(H, B_t \setminus X)$ . That is,  $(S_1 \cup S_2, L_1 \oplus L_2)$  is a partial solution with respect to  $c[t, M]$  and  $\mathcal{X}$ , and thus we have  $\mathcal{X} \in c[t, M]$ .  $\diamond$

We update  $r[t, M]$  as follows. Set  $\mathcal{K} := \emptyset$ . We fix integers  $i_1, i_2$  with  $i_1 + i_2 = i$ ,  $(g, h_1) \in \mathcal{F}(t_1, X, L)$  and  $(g, h_2) \in \mathcal{F}(t_2, X, L)$ . We can check in time  $\mathcal{O}(wd^2)$  the condition that

- for each  $B \in \text{Block}(t, X)$ ,  $h_1(B) \cap h_2(B) = \emptyset$  and  $h(B) = h_1(B) \cup h_2(B)$ , and for  $\ell_1 \in h_1(B)$  and  $\ell_2 \in h_2(B)$ , the vertices with labels  $\ell_1$  and  $\ell_2$  in  $g(B)$  are not adjacent.

If these pairs do not satisfy this condition, then we skip them. We assume that these pairs satisfy this condition. For  $\mathcal{X}_1 \in r[t_1, M_1]$  and  $\mathcal{X}_2 \in r[t_2, M_2]$ , we test whether  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1 \cup \mathcal{X}_2)$  has no cycles and  $\mathcal{X} = \mathcal{X}_1 \uplus \mathcal{X}_2$ . We can check this in time  $\mathcal{O}(w)$ . If they satisfy the two conditions, then we add the partition  $\mathcal{X}$  to the set  $\mathcal{K}$ , and otherwise, we do not add it. After we do this for all possible candidates, we take a representative set of  $\mathcal{K}$  using Proposition 4.3, and assign the resulting set to  $r[t, M]$ . The total running time is  $k \cdot 2^{\mathcal{O}(wd^2)}$  because  $|\mathcal{F}(t_j, X, L)| \leq 2^{\mathcal{O}(wd^2)}$  and  $|r[t_j, M_j]| \leq w \cdot 2^{w-1}$  for each  $j \in \{1, 2\}$ . We have  $|r[t, M]| \leq w \cdot 2^{w-1}$ .

We claim that  $r[t, M] \equiv c[t, M]$ . Let  $\mathcal{X} \in c[t, M]$  and let  $(S, L')$  be a partial solution with respect to  $\mathcal{X}$  and let  $S_{out} \subseteq V(G) \setminus V(G_t)$  where  $G - (S \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . Let  $H_{out} := G - (V(G_t) \setminus B_t) - (X \cup S_{out})$ , and for each  $j \in \{1, 2\}$ , let  $S_j = V(H_j) \cap S$ . Note that every  $(B_t \setminus X)$ -block of  $G - (S \cup X \cup S_{out})$  is chordal.

We first consider  $G - (S \cup X \cup S_{out})$  as the sum  $(H_1, B_t \setminus X) \oplus (H_2 \cup H_{out}, B_t \setminus X)$ . Since  $G - (S \cup X \cup S_{out})$  is chordal, by Proposition 3.7,  $\mathbf{Aux}(H_1, B_t \setminus X) \oplus \mathbf{Aux}(H_2 \cup H_{out}, B_t \setminus X)$  has no cycles. As  $r[t_1, M_1] \equiv c[t_1, M_1]$ , there exists  $\mathcal{Y}_1 \in r[t_1, M_1]$  and a partial solution  $(S'_1, L_1)$  with respect to  $\mathcal{Y}_1$  such that  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{Y}_1) \sim \mathbf{Aux}(G_t - (X \cup S'_1), B_t \setminus X)$ , and

$$\mathbf{Aux}(G_{t_1} - (X \cup S'_1), B_t \setminus X) \oplus \mathbf{Aux}(H_2 \cup H_{out}, B_t \setminus X)$$

has no cycles. By Theorem 5.1,  $G - (S'_1 \cup S_2 \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . Let  $H'_1 := G_{t_1} - (X \cup S'_1)$ . In a similar manner, we consider  $G - (S'_1 \cup S_2 \cup X \cup S_{out})$  as the sum  $(H'_1 \cup H_{out}, B_t \setminus X) \oplus (H_2, B_t \setminus X)$ . Since  $G - (S'_1 \cup S_2 \cup X \cup S_{out})$  is chordal, by Proposition 3.7,  $\mathbf{Aux}(H'_1 \cup H_{out}, B_t \setminus X) \oplus \mathbf{Aux}(H_2, B_t \setminus X)$  has no cycles. As  $r[t_2, M_2] \equiv c[t_2, M_2]$ , there exist  $\mathcal{Y}_2 \in r[t_2, M_2]$  and a partial solution  $(S'_2, L_2)$  with respect to  $\mathcal{Y}_2$  such that  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{Y}_2) \sim \mathbf{Aux}(G_t - (X \cup S'_2), B_t \setminus X)$ , and

$$\mathbf{Aux}(H_1 \cup H_{out}, B_t \setminus X) \oplus \mathbf{Aux}(G_{t_2} - (X \cup S'_2), B_t \setminus X)$$



has no cycles. By Theorem 5.1,  $G - (S'_1 \cup S'_2 \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . Thus the partition  $\mathcal{X}_1 = \mathcal{Y}_1 \uplus \mathcal{Y}_2$  which satisfies  $\mathbf{Inc}(\text{Comp}(t, X), \mathcal{X}_1) \sim \mathbf{Aux}(G_t - (X \cup S'_1 \cup S'_2), B_t \setminus X)$  is added to the set  $\mathcal{K}$ . And there exists  $\mathcal{X}_2 \in r[t, M]$  and a partial solution  $(S'', L''')$  with respect to  $\mathcal{X}_2$  such that  $G - (S'' \cup X \cup S_{out})$  is a  $d$ -labeled  $\mathcal{P}$ -block graph respecting  $(g, h)$ . This shows that  $r[t, M] \equiv c[t, M]$ .

**Total running time.** We denote  $|V(G)|$  by  $n$ . Note that the number of nodes in  $T$  is  $\mathcal{O}(wn)$  by Lemma 2.2. For fixed  $t \in V(T)$ , there are at most  $2^{w+1}$  possible choices for  $X \subseteq B_t$ , and for fixed  $X \subseteq B_t$ , there are at most  $d^{w+1}$  possible functions  $L$ . Furthermore, the size of  $\mathcal{F}(t, X, L)$  is bounded by  $2^{\mathcal{O}(wd^2)}$ . Thus, there are  $\mathcal{O}(n \cdot k \cdot \max(2, d)^{w+1} \cdot 2^{\mathcal{O}(wd^2)})$  tables.

In summary, the algorithm runs in time  $\mathcal{O}(n \cdot k \cdot \max(2, d)^{w+1} \cdot 2^{\mathcal{O}(wd^2)} \cdot k) = 2^{\mathcal{O}(wd^2)} k^2 n$ .  $\square$

We finish this section with a few remarks regarding BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION. For this problem, we think of graphs as labeled graphs where each component consists of vertices with distinct labels from 1 to  $d$ . Let  $\text{Comp}(G, S)$  be the set of connected components of  $G[S]$ . For such a graph  $(G, S)$ , we define a ‘characteristic’ as a pair  $(g, h)$  of functions  $g : \text{Comp}(G, S) \rightarrow \mathcal{U}_d$  and  $h : \text{Comp}(G, S) \rightarrow 2^{[d]}$  satisfying the following, for  $C \in \text{Comp}(G, S)$  and the component  $H$  of  $G$  containing  $C$ ,

- (a) (label-isomorphic condition)  $H$  is partially label-isomorphic to  $g(C)$ ,
- (b) (coincidence condition) for every  $C' \in \text{Comp}(G, S)$  where  $C'$  is contained in  $H$ ,  $g(C') = g(C)$ ,
- (c) (neighborhood condition)  $h(C) = L(N_H(V(C)) \setminus S)$ , and
- (d) (complete condition) for every  $w \in V(H) \setminus S$ ,  $H[N_H[w]]$  is label-isomorphic to  $g(C)[N_{g(C)}[w]]$  where  $w$  is the vertex in  $g(C)$  with label  $L(w)$ .

Then, by following similar, but simpler, arguments, one can also prove that BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION can be solved in time  $2^{\mathcal{O}(wd^2)} k^2 n$ . We omit the details.

**Theorem 1.3.** *Let  $\mathcal{P}$  be a class of graphs that is hereditary, recognizable in polynomial time, and consists of only chordal graphs. Then BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION can be solved in time  $2^{\mathcal{O}(wd^2)} k^2 n$  on graphs with  $n$  vertices and treewidth  $w$ .*

## 6 Lower bound for fixed $d$

We showed that BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION and BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION admit single-exponential time algorithms parameterized by treewidth, whenever  $\mathcal{P}$  is a class of chordal graphs. We now establish that, assuming the ETH, this is no longer the case when  $\mathcal{P}$  contains a graph that is not chordal.

In the  $k \times k$  INDEPENDENT SET problem, one is given a graph  $G = ([k] \times [k], E)$  over the  $k^2$  vertices of a  $k$ -by- $k$  grid. We denote by  $\langle i, j \rangle$  with  $i, j \in [k]$  the vertex of  $G$  in the  $i$ -th row and  $j$ -th column. The goal is to find an independent set of size  $k$  in  $G$  that contains exactly one vertex in each row. The PERMUTATION  $k \times k$  INDEPENDENT SET problem is similar but with the additional constraint that the independent set should also contain exactly one vertex per column.

**Theorem 6.1.** *If  $\mathcal{P}$  contains the cycle graph on  $\ell \geq 4$  vertices, then BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION, or BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION, is not solvable in time  $2^{o(w \log w)} n^{\mathcal{O}(1)}$  on graphs of treewidth at most  $w$  even for fixed  $d = \ell$ , unless the ETH fails.*

*Proof.* We reduce from PERMUTATION  $k \times k$  INDEPENDENT SET which, like PERMUTATION  $k \times k$  CLIQUE, cannot be solved in time  $2^{o(k \log k)} k^{\mathcal{O}(1)}$  unless the ETH fails [13]. Let  $G = ([k] \times [k], E)$  be an instance of PERMUTATION  $k \times k$  INDEPENDENT SET. We assume that  $\forall h, i, j \in [k]$  with  $h \neq i$ ,  $\langle i, j \rangle \langle h, j \rangle \in E$ . Adding these edges does not change the YES- and NO-instances, but has the virtue of making PERMUTATION  $k \times k$  INDEPENDENT SET equivalent to  $k \times k$  INDEPENDENT SET. We also assume that  $\forall h, i, j \in [k]$ ,  $\langle i, j \rangle \langle i, h \rangle \notin E$ , since at most one of  $\langle i, j \rangle$  and  $\langle i, h \rangle$  can be in a given solution. Let  $m := |E| = \mathcal{O}(k^4)$  be the number of edges of  $G$ .

**Outline.** We build two almost identical graphs  $G' = (V', E')$  and  $G'' = (V', E'')$  with treewidth at most  $(3d + 4)k + 6d - 5 = \mathcal{O}(k)$ , and  $((3d - 2)k^2 + 2k)m$  vertices, such that the following three conditions are equivalent:

1.  $G$  has an independent set of size  $k$  with one vertex per row of  $G$ .
2. There is a set  $S \subseteq V'$  of size at most  $(3d - 2)k(k - 1)m$  such that each connected component of  $G' - S$  has size at most  $d$ .
3. There is a set  $S \subseteq V'$  of size at most  $(3d - 2)k(k - 1)m$  such that each block of  $G'' - S$  has size at most  $d$ .

The overall construction of  $G'$  and  $G''$  will display  $m$  *almost* copies of the encoding of an *edgeless*  $G$  arranged in a cycle. Each copy embeds one distinct edge of  $G$ . The point of having the information of  $G$  distilled edge by edge in  $G'$  and  $G''$  is to control the treewidth. This general idea originates from a paper of Lokshtanov et al. [11].

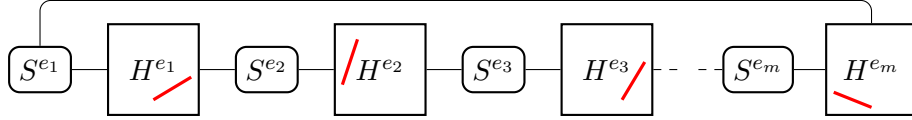


Figure 3: A high-level schematic of  $G'$  and  $G''$ . The  $H^{e_i}$ s only differ by a constant number of edges (in red/light gray) that encode their edge  $e_i$  of  $G$ .

**Construction.** We first describe  $G'$ . As a slight abuse of notation, a gadget (and, more generally, a subpart of the construction) may refer to either a subset of vertices or to an induced subgraph. For each  $e = \langle i^e, j^e \rangle \langle i'^e, j'^e \rangle \in E$ , we detail the internal construction of  $H^e$  and  $S^e$  of Fig. 3 and how they are linked to one another. Each vertex  $v = \langle i, j \rangle$  of  $G$  is represented by a gadget  $H^e(v)$  on  $3d - 2$  vertices in  $G'$ : a path on  $d - 3$  vertices whose endpoints are  $v_{-a}^e$  and  $v_{-b}^e$ , an isolated vertex  $v_+^e$ , and two disjoint cycles of length  $d$ . Observe that if  $d = 4$ , then  $v_{-a}^e$  and  $v_{-b}^e$  is the same vertex. We add all the edges between  $H^e(\langle i, j \rangle)$  and  $H^e(\langle i', j' \rangle)$  for  $i, j, j' \in [k]$  with  $j \neq j'$ . We also add all the edges between  $H^e(\langle i^e, j^e \rangle)$  and  $H^e(\langle i'^e, j'^e \rangle)$ . We call  $H^e$  the graph induced by the union of every  $H^e(v)$ , for  $v \in V(G)$ . The *row/column selector* gadget  $S^e$  consists of a set  $S_r^e$  of  $k$  vertices with one vertex  $r_i^e$  for each row index  $i \in [k]$ , and a set  $S_c^e$  of  $k$  vertices with one vertex  $c_j^e$  for each column index  $j \in [k]$ . The gadget  $S^e$  forms an independent set of size  $2k$ . We arbitrarily number the edges of  $G$ :  $e_1, e_2, \dots, e_m$ . For each  $h \in [m]$  and  $v = \langle i, j \rangle \in V$ , we link  $v_{-a}^{e_h}$  to  $r_i^{e_h}$  (the row index of  $v$ ) and  $v_{-b}^{e_h}$  to  $c_j^{e_h}$  (the column index of  $v$ ). We also link, for every  $h \in [m - 1]$ ,  $v_+^{e_h}$  to  $r_i^{e_{h+1}}$  and to  $c_j^{e_{h+1}}$ , and  $v_+^{e_m}$  to  $r_i^{e_1}$  and to  $c_j^{e_1}$ . That concludes the construction (see Fig. 4). To obtain  $G''$  from  $G'$ , we add the edges  $c_j^{e_h} c_{j+1}^{e_h}$  for every  $h \in [m]$  and  $j \in [k - 1]$ . We ask for a deletion set  $S$  of size  $s := (3d - 2)k(k - 1)m$ .

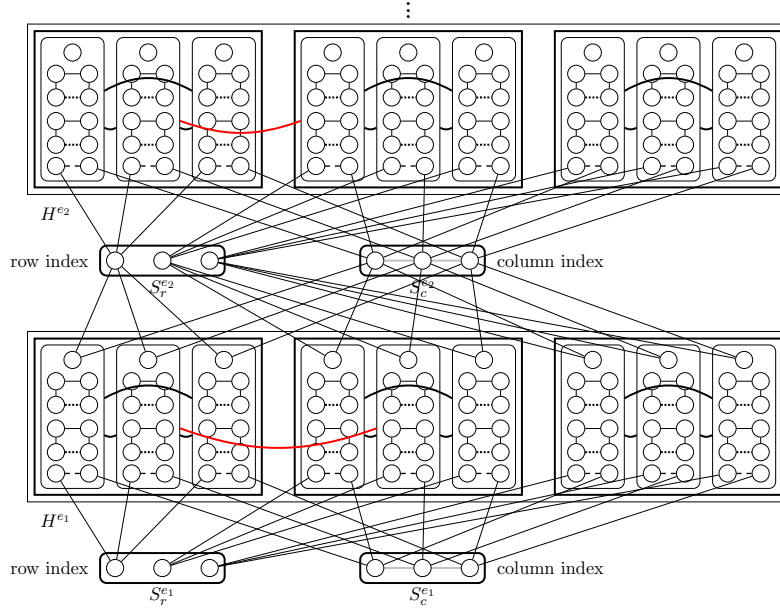


Figure 4: The overall picture of  $G'$  and  $G''$  with  $k = 3$ . Dotted edges are subdivided  $d - 4$  times. In particular, if  $d = 4$ , they are simply edges. Dashed edges are subdivided  $d - 5$  times. In particular, if  $d = 4$ , the two endpoints are in fact a single vertex. Edges between two boxes link each vertex of one box to each vertex of the other box. The gray edges in the column selectors  $S_c^{e_h}$  are only present in  $G''$ .

**Treewidth of  $G'$  and  $G''$ .** We claim that the pathwidth, and hence treewidth, of  $G'$  and  $G''$  are bounded by  $(3d+4)k+6d-5$ . For any edge  $e \in E$ , we set  $H(e) := H^e(\langle i^e, j^e \rangle) \cup H^e(\langle i'^e, j'^e \rangle)$ . For any  $i \in [m-1]$ , we set  $\tilde{S}_i := S^{e_1} \cup S^{e_i} \cup S^{e_{i+1}}$ , and  $\tilde{S}_m := S^{e_1} \cup S^{e_m}$ . For each  $e \in E$ , and  $i \in [k]$ ,  $H^e(i)$  denotes the union of the  $H^e(v)$  for all vertices  $v$  of the  $i$ -th row. Here is a path decomposition of  $G'$  and  $G''$  where the bags contain no more than  $(3d+4)k+6d-4$  vertices:

$$\begin{aligned} \tilde{S}_1 \cup H(e_1) \cup H^{e_1}(1) &\rightarrow \tilde{S}_1 \cup H(e_1) \cup H^{e_1}(2) \rightarrow \dots \rightarrow \tilde{S}_1 \cup H(e_1) \cup H^{e_1}(k) \rightarrow \\ \tilde{S}_2 \cup H(e_2) \cup H^{e_2}(1) &\rightarrow \tilde{S}_2 \cup H(e_2) \cup H^{e_2}(2) \rightarrow \dots \rightarrow \tilde{S}_2 \cup H(e_2) \cup H^{e_2}(k) \rightarrow \\ &\vdots \\ \tilde{S}_m \cup H(e_m) \cup H^{e_m}(1) &\rightarrow \tilde{S}_m \cup H(e_m) \cup H^{e_m}(2) \rightarrow \dots \rightarrow \tilde{S}_m \cup H(e_m) \cup H^{e_m}(k). \end{aligned}$$

As, for any  $h \in [m]$ ,  $|\tilde{S}_h| \leq 6k$ ,  $|H(e_h)| = 2(3d-2)$ , and  $|H^{e_h}(i)| \leq (3d-2)k$  for any  $i \in [k]$ , the size of a bag is bounded by  $\max_{h \in [m], i \in [k]} |\tilde{S}_h \cup H(e_h) \cup H^{e_h}(i)| \leq 6k + 2(3d-2) + (3d-2)k = (3d+4)k + 6d - 4$ .

**Corectness.** We first show  $1 \Rightarrow 2$ . Let us assume that there is an independent set  $I := \{v_1 = \langle 1, j_1 \rangle, v_2 = \langle 2, j_2 \rangle, \dots, v_k = \langle k, j_k \rangle\}$  in  $G$ . We define the deletion set  $S \subseteq V'$  as follows. For each  $e \in E$  and  $i \in [k]$ , we delete all of  $H^e(i)$  except  $H^e(v_i)$ . The cardinality of  $S$  adds up to a total of  $(|H^e(i)| - |H^e(v_i)|)mk = ((3d-2)k - 3d+2)mk = (3d-2)k(k-1)m = s$  vertices. We claim that all the connected components of  $G' - S$  are isomorphic to  $C_d$ . First, we observe that the  $C_d$ s inside any  $H^e(v_i)$ , for  $e \in E$  and  $i \in [k]$ , are isolated in  $G' - S$ . Indeed,  $H^e(v_i)$  is the only remaining  $H^e(v)$  from  $H^e(i)$ . So, it might only be linked to  $H^e(v_j)$  with some  $j \neq i \in [k]$ . But this would imply that  $v_i v_j \in E$ , contradicting that  $I$  is an independent set. Besides those  $C_d$ s contained in the  $H^e(v_i)$ s, we claim that the rest of  $G' - S$  is  $mk$  disjoint  $C_d$ s formed with the vertices  $v_{p+}^{e_{h-1}}, r_p^{e_h}, c_{j_p}^{e_h}$ , and the path  $P_{v_p}^{e_h}$  between  $v_{p-a}^{e_h}$  and  $v_{p-b}^{e_h}$ , for any  $h \in [m]$  and  $p \in [k]$  (with the convention that  $e_0 = e_m$ ). Indeed, let us recall that  $\{j_1, j_2, \dots, j_k\} = [k]$ . Therefore,  $\{v_{p+}^{e_{h-1}}, r_p^{e_h}, c_{j_p}^{e_h}\} \cup P_{v_p}^{e_h}$  is a family of  $mk$  pairwise disjoint sets of size  $d$ . The vertices  $r_p^{e_h}$  and  $c_{j_p}^{e_h}$  have degree 2 in  $G' - S$  since  $I$  contains only one vertex in the  $p$ -th row of  $G$ , and  $I$  contains only one vertex in the  $j_p$ -th column; and in both cases this vertex is  $v_p$ . The vertex  $v_{p+}^{e_{h-1}}$  and the vertices of  $P_{v_p}^{e_h}$  also have degree 2 in  $G' - S$ . Therefore,  $G' - S$  is a disjoint union of  $C_d$ s. The implication  $1 \Rightarrow 3$  is derived similarly. We now claim that, with the same deletion set  $S$ , all the blocks of  $G'' - S$  are isomorphic to  $C_d$  or  $K_2$ . As  $\mathcal{P}$  is a hereditary class that contains the induced cycle of length  $d \geq 4$ , it holds that  $K_2 \in \mathcal{P}$ . We still have the property that the  $C_d$ s within any  $H^e(v_i)$  are isolated in  $G'' - S$ . Now, the slight difference is that  $\{v_{p+}^{e_{h-1}}, r_p^{e_h}, c_{j_p}^{e_h}\} \cup P_{v_p}^{e_h}$  induces  $m$  disjoint  $\mathcal{C}_{k,d}$ s in  $G'' - S$ , where  $\mathcal{C}_{k,d}$  is the graph obtained by linking each of the  $k$  vertices of a path to the two endpoints of a path on  $d-1$  vertices. Informally,  $\mathcal{C}_{k,d}$  corresponds to  $k$   $C_d$ s attached to different vertices of a path on  $k$  vertices. In this case, the path consists of the vertices  $c_1^{e_h}, c_2^{e_h}, \dots, c_k^{e_h}$ . Finally, we observe that the blocks of  $\mathcal{C}_{k,d}$  are  $k$   $C_d$ s and  $k-1$   $K_2$ s.

We now show that  $2 \Rightarrow 1$  and  $3 \Rightarrow 1$ . We assume that there is a set  $S \subseteq V'$  of size at most  $s$  such that all the blocks of  $G'' - S$  (resp.  $G' - S$ ) have size at most  $d$ . We note that this corresponds to assuming 3 (resp. a weaker assumption than 2). The first property we show on  $S$  is that, for any  $e \in E$  and  $i \in [k]$ ,  $|H^e(i) \cap S| \geq (3d-2)(k-1)$ . In other words, there are at most  $3d-2$  vertices of  $H^e(i)$  remaining in  $G'' - S$  (or  $G' - S$ ). Assume, for the sake of contradiction, that  $H^e(i) - S$  contains at least  $3d-1$  vertices. Observe that  $H^e(i) - S$  cannot contain at least one vertex from three distinct  $H^e(u)$ ,  $H^e(v)$ , and  $H^e(w)$  (with  $u, v$  and  $w$  in the  $i$ -th row of  $G$ ), since then  $H^e(i) - S$  would be 2-connected (and of size  $> d$ ). For the same reason,  $H^e(i) - S$  cannot contain at least two vertices in  $H^e(u)$  and at least two vertices in another  $H^e(v)$ . Therefore, the only way of fitting  $3d-1$  vertices

in  $H^e(i) - S$  is the  $3d - 2$  vertices of an  $H^e(u)$  plus one vertex from some other  $H^e(v)$ . But then, this vertex of  $H^e(v)$  would form, together with one  $C_d$  of  $H^e(u)$ , a 2-connected subgraph of  $G'' - S$  (or  $G' - S$ ) of size  $d + 1$ . Now, we know that  $|H^e(i) \cap S| \geq (3d - 2)(k - 1)$ . As there are precisely  $mk$  sets  $H^e(i)$  in  $G'$  (and they are disjoint), it further holds that  $|H^e(i) \cap S| = (3d - 2)(k - 1)$ , since otherwise  $S$  would contain strictly more than  $s = (3d - 2)k(k - 1)m$  vertices. Thus,  $H^e(i) - S$  contains exactly  $3d - 2$  vertices. By the previous remarks,  $H^e(i) - S$  can only consist of the  $3d - 2$  vertices of the same  $H^e(u)$  or  $3d - 3$  vertices of  $H^e(u)$  plus one vertex from another  $H^e(v)$ . In fact, the latter case is not possible, since the vertex of  $H^e(v)$  would form, with at least one remaining  $C_d$  of the  $3d - 3$  vertices of  $H^e(u)$ , a 2-connected subgraph of  $G'' - S$  (or  $G' - S$ ) of size  $d + 1$ . Note that this is why we needed two disjoint  $C_d$ s in the construction instead of just one. So far, we have proved that, assuming 2 or 3, for any  $e \in E$  and  $i \in [k]$ ,  $H^e(i) \cap S = H^e(v_{i,e})$  for some vertex  $v_{i,e}$  of the  $i$ -th row of  $G$ , and for any  $e \in E$ ,  $S^e \cap S = \emptyset$ .

The second part of the proof consists of showing that  $v_{i,e}$  does not depend on  $e$ . Formally, we want to show that there is a  $v_i$  such that, for any  $e \in E$ ,  $v_{i,e} = v_i$ . Observe that it is enough to derive that, for any  $h \in [m]$ ,  $v_{i,e_h} = v_{i,e_{h+1}}$  (with  $e_{m+1} = e_1$ ). Let  $j \in [k]$  (resp.  $j' \in [k]$ ) be the column of  $v_{i,e_h}$  (resp.  $v_{i,e_{h+1}}$ ) in  $G$ . We first assume 2. For any  $h \in [m]$ ,  $v_{i,e_h}^{e_h}, r_i^{e_{h+1}}, c_j^{e_{h+1}}, c_j^{e_{h+1}}$  plus the path  $P_{v_{i,e_{h+1}}}^{e_{h+1}}$  (between  $v_{i,e_{h+1}-a}^{e_{h+1}}$  and  $v_{i,e_{h+1}-b}^{e_{h+1}}$ ) induces a path (in particular, a connected subgraph) of size  $d + 1$  in  $G'' - S$ , unless  $j = j'$  (with  $e_{m+1} = e_1$ ). Therefore,  $j = j'$ . As  $v_{i,e_h}$  and  $v_{i,e_{h+1}}$  have the same column  $j$  and the same row  $i$  in  $G$ ,  $v_{i,e_h} = v_{i,e_{h+1}}$ .

Now, we assume 3. For any  $h \in [m]$ ,  $v_{i,e_h}^{e_h}, r_i^{e_{h+1}}, v_{i,e_{h+1}-a}^{e_{h+1}}, c_{j'}^{e_{h+1}}, c_{j'+1}^{e_{h+1}}, \dots, c_{j-1}^{e_{h+1}}, c_j^{e_{h+1}}$  if  $j \geq j'$  (resp.  $c_{j'-1}^{e_{h+1}}, \dots, c_{j+1}^{e_{h+1}}, c_j^{e_{h+1}}$  if  $j \leq j'$ ) plus the path between  $v_{i,e_{h+1}-a}^{e_{h+1}}$  and  $v_{i,e_{h+1}-b}^{e_{h+1}}$  induces a cycle (that is, a 2-connected subgraph) of length at least  $d + 1$  in  $G'' - S$ , unless  $j = j'$  (with  $e_{m+1} = e_1$ ). Again,  $j = j'$ ; and the vertices  $v_{i,e_h}$  and  $v_{i,e_{h+1}}$  have the same column and the same row in  $G$ , which implies that  $v_{i,e_h} = v_{i,e_{h+1}}$ . In both cases (2 or 3), we can now safely define  $v_i := v_{i,e}$ .

We finally claim that  $\{v_1, v_2, \dots, v_k\}$  is an independent set in  $G$  (and for each  $i \in [k]$ ,  $v_i$  is in the  $i$ -th row). Indeed, if there were an edge  $e = v_i v_j \in E$  for some  $i \neq j \in [k]$ , then  $H^e(v_i) \cup H^e(v_j)$  would induce a 2-connected subgraph of size  $2(3d - 2) > d$  (since  $d \geq 4$ ) in  $G'' - S$  (or  $G' - S$ ).

That finishes the proof that  $1 \Leftrightarrow 2 \Leftrightarrow 3$ . Therefore, for any fixed integer  $d \geq 4$ , an algorithm running in time  $2^{o(w \log w)} |V'|^{\mathcal{O}(1)}$  for either BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION or BOUNDED  $\mathcal{P}$ -BLOCK VERTEX DELETION on graphs of treewidth  $w$  with  $C_d \in \mathcal{P}$  would also solve PERMUTATION  $k \times k$  INDEPENDENT SET in time

$$2^{o(((3d+4)k+6d-5) \log((3d+4)k+6d-5))} (((3d-2)k^2 + 2k)m)^{\mathcal{O}(1)} = 2^{o(k \log k)} k^{\mathcal{O}(1)},$$

which contradicts the ETH. □

## 7 Hardness and lower bounds, when $d$ is not fixed

In this section, we prove Theorem 1.5. Our first reduction is from the following problem:

**MULTICOLORED CLIQUE**

**Parameter:**  $k$

**Input:** A graph  $G$ , a positive integer  $k$ , and a partition  $(V_1, V_2, \dots, V_k)$  of  $V(G)$ .

**Question:** Is there a  $k$ -clique  $X$  of  $G$  such that  $|X \cap V_i| = 1$  for each  $i \in [k]$ ?

We call a set  $V_i$ , for some  $i \in [k]$ , a *color class*. The problem MULTICOLORED CLIQUE is known to be  $W[1]$ -complete (see, for example, [5]), and it is clear that this remains true under the assumption that there are no edges between vertices of the same color class. Moreover, we may assume that each color class has the same size, and between every distinct pair of color classes we have the same number of edges [9]. We say that  $X \subseteq V(G)$  is a *multicolored  $k$ -clique* if  $X$  is a  $k$ -clique such that  $|X \cap V_i| = 1$  for each  $i \in [k]$ .

**Theorem 7.1.** BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION is  $W[1]$ -hard parameterized by the combined parameter  $(w, k)$ , when  $\mathcal{P}$  contains all chordal graphs.

Before proving this theorem, we describe the reduction used in the proof. Given an instance  $(G, k, (V_1, \dots, V_k))$  of MULTICOLORED CLIQUE, where each color class has size  $t$ , we construct a graph  $G'$  such that  $G$  has a multicolored  $k$ -clique if and only if there exists a set  $S \subseteq V(G')$  of size at most  $k'$  such that each connected component of  $G' - S$  consists of at most  $d$  vertices, where  $k' = 3\binom{k+1}{2} - 6$  and  $d = 3t^2 + 3t + 3$ , and the treewidth of  $G'$  is bounded above by  $54k - 69$ . We may assume that  $k \geq 2$ .

Let  $V_i = \{v_i^1, v_i^2, \dots, v_i^t\}$ , for each  $i \in [k]$ . For  $i, j \in [k]$  with  $i < j$ , we denote the set of edges in  $G[V_i \cup V_j]$  by  $E_{i,j}$ , and we may assume that  $|E_{i,j}| = p$ , say. We construct  $G'$  from several gadgets; namely, an “edge-encoding gadget”  $G_{i,j}$  for each  $i, j \in [k]$  with  $i < j$ , which represents the set  $E_{i,j}$ , linked together by copies of one of the “propagator gadgets”,  $H_i$  or  $\tilde{H}_i$ , which collectively represent the color class  $V_i$  for some  $i \in [k]$ . We also have a gadget  $G_{i,i}$ , for each  $i \in [2, k-2]$ , which ensures that the vertex selection in the  $H_i$  gadgets also propagates to the  $\tilde{H}_i$  gadgets.

Each gadget encodes a sequence of integers  $X = \langle x_0, x_1, \dots, x_{z+1} \rangle$ , where  $x_0 \geq 3$ , and  $x_s - x_{s-1} \geq 3$  for each  $s \in [z+1]$ . We denote such a gadget  $G(X)$  and call it a *gadget of  $G'$  of order  $z$* . It is constructed as follows. First, set

$$(d_0, d_1, d_2, \dots, d_z) := (x_0, x_1 - x_0, x_2 - x_1, \dots, x_z - x_{z-1}, x_{z+1} - x_z).$$

Note that  $d_q \geq 3$  for every  $q \in [0, z]$ . For each  $q \in [0, z]$ , we now define a graph  $P_q$  which resembles a “thickened path”. For  $q \in [1, z-1]$ , let  $P_q$  be the graph on the vertex set  $\{w_{q,1}, w_{q,2}, \dots, w_{q,d_q-1}\}$  with edges between distinct  $w_{q,d}$  and  $w_{q,d'}$  if and only if  $|d - d'| \in [2]$ . For  $q \in \{0, z\}$ , let  $P_q$  be the graph on the vertex set  $\{w_{q,1}, w_{q,2}, \dots, w_{q,d_q}\}$  with edges between distinct  $w_{q,d}$  and  $w_{q,d'}$  if and only if  $|d - d'| \in [3]$ . For each  $q \in [z]$ , we add a vertex  $u_q$  adjacent to  $w_{q-1,1}$ ,  $w_{q-1,2}$ ,  $w_{q,1}$ , and  $w_{q,2}$ . The resulting graph  $G(X)$  consists of  $(\sum_{q \in [z]} d_q) + 1 = x_{z+1} + 1$  vertices, and, for  $q \in [z]$ , the graph obtained by deleting  $u_q$  has two components: one of size  $x_q$ , and the other of size  $x_{z+1} - x_q$ . Let  $B := \{w_{0,1}, w_{0,2}, w_{0,3}\}$  and  $D := \{w_{z,1}, w_{z,2}, w_{z,3}\}$ . Since we will use several copies of this gadget, we usually refer to  $P_q$  as  $P_q(G(X))$ , a vertex  $v \in V(G(X))$  as  $v(G(X))$ , and  $B$  or  $D$  as  $B(G(X))$  or  $D(G(X))$ , respectively; but we sometimes omit the “ $(G(X))$ ” when there is no ambiguity.

We now describe the *edge encoding gadget*  $G_{i,j}$ , for some  $i, j \in [k]$  with  $i < j$ ; an example is given in Fig. 5a. We can uniquely describe an edge between a vertex in  $V_i$  and a vertex in  $V_j$  by an ordered pair  $(a, b)$ , representing the edge  $v_i^a v_j^b$ , where  $a, b \in [t]$ . We define an injective function  $\phi$  from such a pair to an integer in  $\{3, 6, \dots, 3t^2\}$ , as given by  $(a, b) \mapsto 3t(a-1) + 3b$ . Thus, the set  $\{\phi(a, b) : v_i^a v_j^b \in E_{i,j}\}$  uniquely describes the set  $E_{i,j}$ . Let  $(f_{i,j}^0, f_{i,j}^1, \dots, f_{i,j}^p)$  be the sequence obtained after ordering the elements of this set in increasing order, and let  $f_{i,j}^{p+1} = 3t^2 + 3$ . Note that  $f_{i,j}^0 \geq 3$ , and  $f_{i,j}^q - f_{i,j}^{q-1} \geq 3$  for each  $q \in [p+1]$ . Finally, we set  $G_{i,j} := G(\langle f_{i,j}^0, f_{i,j}^1, \dots, f_{i,j}^{p+1} \rangle)$ .

We define the *propagator gadgets* as  $H_i := G(\langle 3, 6, \dots, 3(t+1) \rangle)$  and  $\tilde{H}_i := G(\langle 3t, 6t, \dots, 3(t+1)t \rangle)$ ; see Figs. 5b and 5c. Note that these gadgets have size  $3(t+1) + 1$

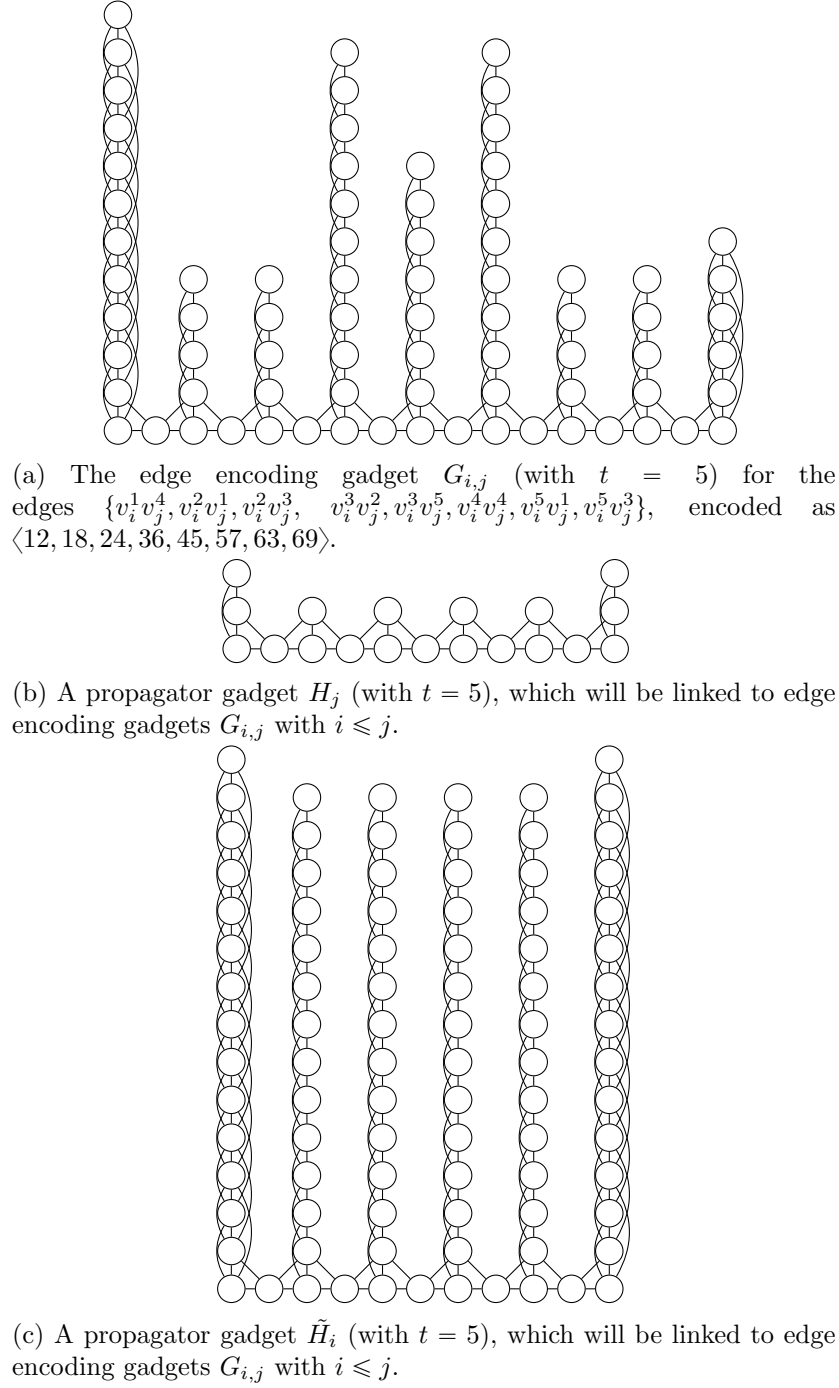


Figure 5: The different uses of the gadgets: the edge encoding gadget and the two kinds of propagator gadgets.

and  $3t(t+1)+1$ , respectively. For each color class  $V_i$ , where  $i \in [2, k-1]$ , we will take  $i$  copies of the gadget  $H_i$ , and  $k-i+1$  copies of  $\tilde{H}_i$ ; whereas for  $i=1$  (or  $i=k$ ), we take  $k-1$  copies of  $\tilde{H}_i$  (or  $H_i$ , respectively) only. Let  $\mathcal{H}_i$  denote the set containing the copies of  $H_i$ , and let  $\tilde{\mathcal{H}}_i$  denote the copies of  $\tilde{H}_i$ . Note that  $|\mathcal{H}_i \cup \tilde{\mathcal{H}}_i| = k+1$  when  $i \in [2, k-1]$ , and  $|\mathcal{H}_i \cup \tilde{\mathcal{H}}_i| = k-1$  when  $i \in \{1, k\}$ .

Finally, for each  $i \in [2, k-2]$ , we have a special gadget  $G_{i,i} := G(\langle \phi(1,1), \phi(2,2), \dots, \phi(t,t) \rangle)$ . Intuitively, this gadget is used to ensure the vertex selected in each  $H_i \in \mathcal{H}_i$  is the same as in each  $\tilde{H}_i \in \tilde{\mathcal{H}}_i$ . However, we also consider  $G_{i,i}$  an edge encoding gadget, since it is treated as one in the construction.

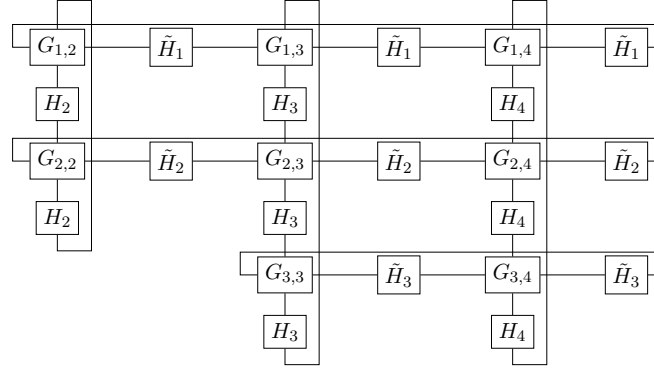


Figure 6: The overall picture with  $k = 4$ .

In order to describe how these gadgets are joined together in  $G'$ , as shown in Fig. 6, we require some terminology. Given some  $G_{i,j}$  and  $G_{i',j'}$  with  $i, j, j' \in [k]$ , we say we *connect*  $G_{i,j}$  to  $G_{i',j'}$  *using*  $\tilde{H}_i$  to describe adding all nine edges between  $D(G_{i,j})$  and  $B(\tilde{H}_i)$ , and all nine edges between  $D(\tilde{H}_i)$  and  $B(G_{i',j'})$ . In this case, we also say  $\tilde{H}_i$  *connects from*  $G_{i,j}$  and *connects to*  $G_{i',j'}$ . Given some  $G_{i,j}$  and  $G_{i',j}$  with  $i, i', j \in [k]$ , the operation of *connecting*  $G_{i,j}$  to  $G_{i',j}$  *using*  $H_j$  is defined analogously. We give the following cyclic ordering to the edge encoding gadgets:  $(G_{1,2}, G_{1,3}, \dots, G_{1,k}, G_{2,2}, G_{2,3}, \dots, G_{2,k}, \dots, G_{k-1,k-1}, G_{k-1,k})$ . For each  $G_{i,j}$ , we connect this gadget to the next gadget  $G_{i,j'}$  in the cyclic ordering that matches on the first index using one of the copies of  $\tilde{H}_i$ , and also connect it to the next gadget  $G_{i',j}$  in the ordering that matches on the second index using one of the copies of  $H_j$ . For example, we connect  $G_{1,3}$  to  $G_{1,4}$  using a copy of  $\tilde{H}_1$ , and connect  $G_{1,3}$  to  $G_{2,3}$  using a copy of  $H_3$ . This completes the construction.

*Proof of Theorem 7.1.* Observe that each vertex  $v \in V(G')$  is contained in precisely one gadget, and so each vertex of  $G'$  inherits either a ‘ $u$ ’ label or a ‘ $w$ ’ label from its gadget. In what follows, whenever we refer to an edge encoding gadget  $G_{i,j}$ , or a propagator gadget  $\tilde{H}_i$  or  $H_j$ , it is for some  $i \in [1, k-1]$  and  $j \in [2, k]$  with  $i \leq j$ .

**Treewidth.** We now describe a path decomposition of  $G'$  that illustrates that its pathwidth, and hence treewidth, is at most  $54k - 69$ .

First, observe that for a gadget  $H := G(\langle x_0, x_1, \dots, x_{z+1} \rangle)$ , there is a path decomposition where each bag has size at most 4. By adding  $B(H) \cup D(H)$  to every bag, we obtain a path decomposition where each bag has size at most 10; we denote this path decomposition by  $\mathcal{P}(H)$ . Note that  $H$  is only linked to other gadgets in  $G'$  by edges with one end in either  $B(H)$  or  $D(H)$ .

Recall that the edge encoding gadgets are joined together using propagator gadgets with respect



to the cyclic ordering

$$(G_{1,2}, G_{1,3}, \dots, G_{1,k}, G_{2,2}, G_{2,3}, \dots, G_{2,k}, \dots, G_{k-1,k-1}, G_{k-1,k}).$$

Consider an auxiliary multigraph  $F$  on the vertex set  $\{G_{i,j} : i \in [1, k-1], j \in [2, k], i \leq j\}$  where there is an edge between  $G_{i,j}, G_{i',j'} \in V(F)$  whenever the gadget  $G_{i,j}$  is connected to  $G_{i',j'}$  using some propagator gadget in  $G'$ . (Formally, there is an edge for  $i = i'$  and  $|j - j'| \in \{1, k - i, k - 2\}$ , or  $j = j'$  and  $|i - i'| \in \{1, j - 1, k - 2\}$ .)

We now show that  $F$  has pathwidth at most  $3k - 5$ . Let  $\mathcal{G}_1 = \{G_{1,j} : j \in [2, k]\}$  and, for  $i \in [2, k-1]$ , let  $\mathcal{G}_i = \{G_{i,j} : j \in [i, k]\}$ . Then  $(\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3, \mathcal{G}_1 \cup \mathcal{G}_3 \cup \mathcal{G}_4, \dots, \mathcal{G}_1 \cup \mathcal{G}_{k-2} \cup \mathcal{G}_{k-1})$  is a path decomposition for  $F$  where the largest bag, the first one, has size  $3k - 4$ . We denote this path decomposition  $\mathcal{P}(F)$ .

We extend this to a path decomposition of  $G'$  by replacing each bag of  $\mathcal{P}(F)$  with a path, which is in turn constructed from several concatenated “subpaths”, one for each gadget. Suppose, for some  $i, j \in [k]$  with  $i \leq j$ , we have that  $\tilde{H}_i$  and  $H_j$  connect to  $G_{i,j}$  in  $G'$ , and  $\tilde{H}'_i$  and  $H'_j$  connect from  $G_{i,j}$  in  $G'$ ; then we denote  $X_{i,j} = D(\tilde{H}_i) \cup D(H_j) \cup B(G_{i,j}) \cup D(G_{i,j}) \cup B(\tilde{H}'_i) \cup B(H'_j)$ . Let  $Z \subseteq [k] \times [k]$  such that  $\bigcup_{(i,j) \in Z} G_{i,j}$  is a bag of the path decomposition of  $F$ . From this bag, we construct a path where each bag contains  $Q = \bigcup_{(i,j) \in Z} X_{i,j}$ . The subpaths of this path are as follows. For each  $(i, j) \in Z$  we have a subpath obtained from  $\mathcal{P}(G_{i,j})$  by adding  $Q$  to each bag. Every edge of  $F$  is contained in some bag of the path decomposition, and corresponds to a propagator gadget  $H$  of  $G'$ . For each such  $H$ , we have a subpath obtained from  $\mathcal{P}(H)$  by adding  $Q$  to each bag. These subpaths are then concatenated together, end to end, to create the path that replaces the bag  $\bigcup_{(i,j) \in Z} G_{i,j}$  in  $\mathcal{P}(F)$ . After doing this for each bag, we obtain a path decomposition of  $G'$ .

Note that  $|Z| \leq 3k - 4$ , and  $|X_{i,j}| = 18$ , for any  $(i, j) \in Z$ . So  $|Q| \leq 18(3k - 4)$ . A path decomposition  $\mathcal{P}(H)$ , for some gadget  $H$ , has bags with size at most 10, but each bag meets  $Q$  in precisely the elements  $B(H) \cup D(H)$ . So the pathwidth of  $G'$  is at most  $18(3k - 4) + 4 - 1 = 54k - 69$ .

**Correctness ( $\Rightarrow$ ).** First, let  $X$  be a multicolored  $k$ -clique in  $G$ ; we will show that  $G'$  has a set  $S \subseteq V(G')$  such that  $|S| = 3\binom{k+1}{2} - 6$  and each component of  $G' - S$  has at most  $d$  vertices, where  $d = 3t^2 + 3t + 3$ . Let  $\gamma(i)$  be the index of the unique vertex in  $X \cap V_i$  for each  $i \in [k]$ ; that is,  $X \cap V_i = \{v_i^{\gamma(i)}\}$ . For each  $H \in \mathcal{H}_i \cup \tilde{\mathcal{H}}_i$ , we add the vertex  $u_{\gamma(i)}(H)$  to  $S$ ; there are  $(k-2)(k+1) + 2(k-1) = k(k+1) - 4$  such gadgets, so this many vertices are added to  $S$  so far. For each pair  $i, j \in k$  with  $i < j$ , there is some  $q \in [p]$  such that  $\phi(\gamma(i), \gamma(j)) = f_{i,j}^q$ ; we add the vertex  $u_q(G_{i,j})$  to  $S$ . For  $i \in [2, k-2]$ , we also add the vertex  $u_{\gamma(i)}(G_{i,i})$  to  $S$ . Now  $|S| = k(k+1) - 4 + \binom{k}{2} + k - 2 = 3\binom{k+1}{2} - 6$ .

We now consider the size of the components of  $G' - S$ . We first analyze the size of the components of a gadget  $G_{i,j}$ ,  $\tilde{H}_i$  or  $H_j$  after deleting  $S$ . Note that  $S$  meets the vertex set of one of these gadgets in precisely one vertex, and the deletion of this vertex splits the gadget into two components. The two components of  $G_{i,j} - u_q$  have  $f_{i,j}^q = 3t(\gamma(i) - 1) + 3\gamma(j)$  and  $f_{i,j}^{q+1} - f_{i,j}^q = 3t^2 + 3 - (3t(\gamma(i) - 1) + 3\gamma(j))$  vertices. The two components of  $\tilde{H}_i - u_{\gamma(i)}$  have  $3t\gamma(i)$  and  $3t(t+1 - \gamma(i))$  vertices, while the two components of  $H_j - u_{\gamma(j)}$  have  $3\gamma(j)$  and  $3(t+1 - \gamma(j))$  vertices. These gadgets are joined in such a way that the size of a component of  $G' - S$  is

$$\begin{aligned} & [3t(\gamma(i) - 1) + 3\gamma(j)] + 3t(t+1 - \gamma(i)) + 3(t+1 - \gamma(j)) \\ &= 3t^2 + 3t + 3 \\ &= [3t^2 + 3 - (3t(\gamma(i) - 1) + 3\gamma(j))] + 3t\gamma(i) + 3\gamma(j), \end{aligned}$$

as required.

( $\Leftarrow$ ). Suppose  $G'$  has a set  $S \subseteq V(G')$  with  $|S| \leq 3\binom{k+1}{2} - 6$  such that each component of  $G' - S$  has at most  $d$  vertices, where  $d = 3t^2 + 3t + 3$ . We call any such set  $S$  a *solution*.

First, we show, loosely speaking, that we may assume each vertex in  $S$  is a ‘ $u$ ’ vertex of its gadget, not a ‘ $w$ ’ vertex. Let  $H$  be a gadget of  $G'$  of order  $s$ . There are two cases to consider: the first is when, for some  $r \in [1, s-1]$ , we have that  $S \cap V(P_r(H)) \neq \emptyset$ . Suppose  $P_r(H)$  contains a pair of adjacent vertices  $w$  and  $w'$  such that  $\{w, w'\} \cap S \neq \emptyset$ . If  $w \in S$  and  $w' \notin S$ , then, in  $G' - (S \setminus \{w\})$ , only the component containing  $w'$  can have size more than  $d$ , and  $|V(P_r(H))| \leq 3t^2 < d$ , so replacing  $w'$  in  $S$  with  $u_{r-1}(H)$  or  $u_r(H)$  also gives a solution. If  $\{w, w'\} \subseteq S$ , then  $(S \setminus \{w, w'\}) \cup \{u_{r-1}(H), u_r(H)\}$  is also a solution. So we may assume that  $V(P_r(H)) \cap S = \emptyset$  for each  $r \in [1, s-1]$ .

Now we consider the second case; let  $G_{i,j}$  be an edge encoding gadget, let  $H \in \mathcal{H}_i$  and  $\tilde{H} \in \tilde{\mathcal{H}}_j$  connect from  $G_{i,j}$ , and let  $J$  be the set of vertices  $V(P_y(G_{i,j})) \cup V(P_z(H)) \cup V(P_z(\tilde{H}))$ , for  $(y, z) \in \{(p, 0), (0, k+1)\}$ . Observe that  $G'[J]$  is connected and  $|J| \leq d$ ; intuitively, these are the vertices involved in the “join” of multiple gadgets in  $G'$ . We show that if  $J \cap S \neq \emptyset$ , then there is some solution  $S'$  with  $J \cap S' = \emptyset$ . Let  $U := N_{G'}(J)$ , so  $|U| = 3$ . If  $|J \cap S| \geq 3$ , then  $(S \setminus J) \cup U$  is a solution. Moreover, if  $|U \setminus S| \leq |J \cap S|$ , then  $(S \setminus J) \cup U$  is again a solution. Assuming otherwise, we can pick  $U' \subseteq U \setminus S$  such that  $|U'| = |J \cap S|$ . If  $G'[(J \cup U) \setminus S]$  is connected, then  $S' = (S \setminus J) \cup U'$  is a solution. But since  $|J \cap S| \leq 2$ , it follows, by the construction of  $G'$ , that  $G'[J \setminus S]$  is connected. Thus, in the exceptional case, the deletion of  $J \cap S$  disconnects some  $u \in U \setminus S$  from  $G'[J \setminus S]$ . But in this case, if we ensure that  $U'$  is chosen to contain  $u$ , then we still obtain a solution  $S' = (S \setminus J) \cup U'$ .

Next, we claim that each edge encoding gadget  $G_{i,j}$  or propagator gadget  $\tilde{H}_i \in \tilde{\mathcal{H}}_i$ , has at least one vertex in  $S$ . Consider the subgraph  $D_{i,j}$  of  $G'$  induced by  $V(G_{i,j}) \cup V(\tilde{H}_i) \cup V(H_j)$ , where  $\tilde{H}_i$  and  $H_j$  connect from  $G_{i,j}$ . Recall that  $G_{i,j}$  consists of  $3t^2 + 3 + 1$  vertices,  $\tilde{H}_i$  consists of  $3t^2 + 3t + 1$  vertices,  $H_j$  consists of  $3t + 3 + 1$  vertices, and hence  $D_{i,j}$  has size  $2d + 3$ . If  $V(\tilde{H}_i) \cap S$  is empty, then the connected subgraph of  $D_{i,j} - S$  containing  $V(\tilde{H}_i)$  also contains  $P_p(G_{i,j})$ , which has size at least 3, so this connected subgraph contains at least  $3t^2 + 3t + 1 + 3 = d + 1$  vertices; a contradiction. Similarly, if  $V(G_{i,j}) \cap S$  is empty, then the connected subgraph of  $D_{i,j} - S$  containing  $V(G_{i,j})$  also contains at least  $3t$  vertices of  $V(\tilde{H}_i)$ , so at least  $d + 1$  in total; a contradiction. So  $|V(\tilde{H}_i) \cap S|, |V(G_{i,j}) \cap S| \geq 1$ , as claimed.

Now we claim that each connected component of  $G' - S$  has size exactly  $d$ . Pick  $S' \subseteq S$  such that  $|V(G_{i,j}) \cap S'| = 1$  for each edge encoding gadget  $G_{i,j}$ , and  $|V(\tilde{H}_i) \cap S'| = 1$  for each  $\tilde{H}_i \in \tilde{\mathcal{H}}_i$ . So  $|S'| = 2(\binom{k+1}{2} - 2)$ , and  $|S \setminus S'| = \binom{k+1}{2} - 2$ . The graph  $G' - S'$  has  $\binom{k+1}{2} - 2$  components, and the deletion of each vertex in  $S \setminus S'$  further increases the number of components by one. Since  $|V(G')| = (2d+3)(\binom{k+1}{2} - 2)$ , each of the  $\binom{k+1}{2} - 2$  components of  $G' - S'$  has size at least  $2d+1$ , so the remaining  $\binom{k+1}{2} - 2$  vertices in  $S \setminus S'$  must evenly split each of these components into components of size exactly  $d$ , as claimed.

Next we show that each gadget  $H_j \in \mathcal{H}_j$  also has at least one vertex in  $S$ . Suppose we have some  $H_j$  for which  $S \cap V(H_j) = \emptyset$ . We calculate the size, modula 3, of the connected component  $C$  of  $G' - S$  that contains  $H_j$ . Since the size of  $V(C) \cap V(\tilde{H}_i)$  or  $V(C) \cap V(G_{i,j})$  is congruent to 0 (mod 3), and  $|V(H_j)| \equiv 1 \pmod{3}$ , we deduce that  $|V(C)| \equiv 1 \pmod{3}$ ; a contradiction. So  $|S \cap V(H_j)| \geq 1$  for every  $H_j \in \mathcal{H}_j$  with  $j \in [2, k]$ . Since  $|S| = 3\binom{k}{2}$ , it follows that each gadget meets  $S$  in precisely one vertex.

Finally, suppose  $u_q(G_{i,j}) \in S$ , for some  $q \in [p]$ . Then  $\phi(a, b) = f_{i,j}^q$ , for some  $a, b \in [t]$ . Let  $\tilde{H}_i \in \mathcal{H}_i$  and  $H_j \in \mathcal{H}_j$  be the propagators that connect from  $G_{i,j}$ . Now, the connected component

of  $G' - S$  containing  $3t^2 + 3 - (3t(a - 1) + 3b)$  vertices of  $G_{i,j} - u_q$  also contains  $3ta'$  vertices of  $\tilde{H}_i$ , and  $3b'$  vertices of  $H_j$ , for some  $a', b' \in [t]$ . So

$$3t^2 + 3ta' - 3t(a - 1) + 3b' - 3b + 3 = 3t^2 + 3t + 3.$$

Working modula  $t$ , we deduce that  $3(b' - b + 1) \equiv 3 \pmod{t}$ , hence  $b = b'$ . It then follows that  $3t(a' - (a - 1)) = 3t$ , so  $a = a'$ . Thus  $u_a(\tilde{H}_i), u_b(H_j) \in S$ .

On the other hand, if, for some  $a, b \in [t]$  we have  $u_a(\tilde{H}_i), u_b(H_j) \in S$ , where  $\tilde{H}_i$  and  $H_j$  connect to  $G_{i,j}$ , then the component of  $G' - S$  containing vertices from these three gadgets contains  $3t(t + 1 - a)$  vertices from  $\tilde{H}_i$ , as well as  $3(t + 1 - b)$  vertices from  $H_j$ , and  $3t(a' - 1) + 3b'$  from  $G_{i,j}$  for some  $a', b' \in [t]$ . Since this component has a total of  $3t^2 + 3t + 3$  vertices, working modula  $t$  we deduce that  $3b' + 3 - 3b \equiv 3 \pmod{t}$ , so  $b = b'$ . It follows that  $3t(a - a' + 1) = 3t$ , so  $a = a'$ . Thus,  $u_q(G_{i,j}) \in S$  for  $q \in [p]$  such that  $\phi(a, b) = f_{i,j}^q$ .

We deduce that for every  $l \in [k]$ , there exists some  $\gamma(l)$  such that  $V(\tilde{H}) \cap S = \{u_{\gamma(i)}\}$  for every  $\tilde{H} \in \tilde{\mathcal{H}}_i$ ,  $V(H) \cap S = \{u_{\gamma(j)}\}$  for every  $H \in \mathcal{H}_j$ , and  $V(G_{i,j}) \cap S = \{u_q\}$  for  $q \in [p]$  such that  $f_{i,j}^q = \phi(\gamma(i), \gamma(j))$ . It follows that each  $v_i^{\gamma(i)} v_j^{\gamma(j)}$  is an edge of  $G$ , and  $X = \{v_i^{\gamma(i)} : i \in [k]\}$  is a multicolored  $k$ -clique in  $G$ , as required.  $\square$

Theorem 7.1 implies that BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION has no algorithm running in time  $f(w)n^{O(1)}$ , assuming  $\text{FPT} \neq W[1]$ . However, we can say something stronger, assuming the ETH holds. Since, in the parameterized reduction in the previous proof, the treewidth of the reduced instance  $G'$  has linear dependence on  $k$ , a  $f(w)n^{o(w)}$ -time algorithm for this problem would lead to a  $f(k)n^{o(k)}$ -time algorithm for MULTICOLORED CLIQUE. But, assuming the ETH holds, no such algorithm for MULTICOLORED CLIQUE exists [12]. So we have the following:

**Theorem 7.2.** *Unless the ETH fails, there is no  $f(w)n^{o(w)}$ -time algorithm for BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION when  $\mathcal{P}$  contains all chordal graphs.*

Furthermore, Marx [14] showed that, assuming the ETH holds, SUBGRAPH ISOMORPHISM has no  $f(k)n^{o(k/\log k)}$ -time algorithm, where  $k$  is the number of edges in the smaller graph. By reducing from SUBGRAPH ISOMORPHISM, instead of MULTICOLORED CLIQUE, we obtain a lower bound with the combined parameter treewidth and solution size.

**Theorem 7.3.** *Unless the ETH fails, there is no  $f(k')n^{o(k'/\log k')}$ -time algorithm for BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION, where  $k' = w + k$ , when  $\mathcal{P}$  contains all chordal graphs.*

*Proof.* Let  $(G, H)$  be a SUBGRAPH ISOMORPHISM instance where the task is to find if  $G$  has a subgraph isomorphic to  $H$ . Let  $k := |V(H)|$  and  $t := |V(G)|$ , and suppose  $V(G) = \{v^a : a \in [t]\}$  and  $V(H) = \{v_i : i \in [k]\}$ . Let  $V_i = \{v_i^a : a \in [t]\}$  for each  $i \in [k]$ , and let  $G^+$  be the graph on the vertex set  $\bigcup_{i \in [k]} V_i$  with an edge  $v_i^a v_j^b$  if and only if  $i \neq j$  and  $v^a v^b$  is an edge of  $G$ . Now the task is to select  $|E(H)|$  edges of  $G^+$  that induce a *multicolored* subgraph of  $G^+$ ; that is, the vertex set of this edge-induced subgraph meets each  $V_i$  in exactly one vertex.

We construct  $G'$  from  $G^+$  using a similar construction as in the proof of Theorem 7.1, but we only have an edge encoding gadget  $G_{i,j}$  for  $1 \leq i < j \leq k$  when  $v_i v_j$  is an edge in  $H$ . More specifically, we take the subsequence of  $(G_{1,2}, G_{1,3}, \dots, G_{1,k}, G_{2,2}, G_{2,3}, \dots, G_{2,k}, \dots, G_{k-1,k-1}, G_{k-1,k})$  consisting of each  $G_{i,j}$  for which  $v_i v_j \in E(H)$ , as well as  $G_{i,i}$  for all  $i \in [2, k - 1]$ , and, as before, connect each  $G_{i,j}$  to the next  $G_{i,j'}$  in the cyclic ordering that matches on the first index using a copy of  $\tilde{H}_i$ , and

also connect it to the next gadget  $G_{i',j}$  in the ordering that matches on the second index using a copy of  $H_j$ . Note that  $p = |E_{i,j}| = 2|E(G)|$ .

By a routine adaptation of Theorem 7.1, it is easy to see that  $\text{tw}(G') = \mathcal{O}(k)$ , and that  $G$  has a subgraph isomorphic to  $H$  if and only if  $G'$  has a set  $S \subseteq V(G')$  of size at most  $k'$  such that each connected component of  $G' - S$  has size at most  $d$ . Now the parameter in the reduced instance is  $k'' := \text{tw}(G') + k' = \mathcal{O}(|V(H)|) + \mathcal{O}(|V(H)|^2) = \mathcal{O}(|E(H)|)$ . Thus, an  $f(k'')n^{\mathcal{O}(k''/\log k'')}$ -time algorithm for BOUNDED  $\mathcal{P}$ -COMPONENT VERTEX DELETION would lead to an algorithm for SUBGRAPH ISOMORPHISM running in time  $f(|E(H)|)n^{\mathcal{O}(|E(H)|/\log |E(H)|)}$ . But there is no algorithm for SUBGRAPH ISOMORPHISM with this running time unless the ETH fails [14].  $\square$

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